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② Let R be a commutative ring with identity, and let I and J be ideals in R. Recall that $I+J=\{r+r' : r\in I,\ r'\in J\}$, and IJ is the ideal generated by all products rr' with $r\in I$ and $r'\in J$.

(a) Prove that if $I+J=R$, then $IJ = I\cap J$.

Pf: First we will show that $IJ\subset I\cap J$:
We know that $IJ\subset I$ by defn. since I is an ideal and likewise $IJ\subset J$ by defn. since J is an ideal.
Since $IJ\subset I$ and $IJ\subset J$, we have that $IJ\subset I\cap J$. ✓

Next we will show that $I\cap J\subset IJ$:
Let $z\in I\cap J$.
Since $I+J=R$, for $x\in I, y\in J$, we can write $x+y=1$.
If we can show that $z=z(x+y)\in IJ$, then we are done.
$z\in I\cap J \Rightarrow z\in I$ and $z\in J$.
Observe that $z(x+y) = zx+zy \in IJ$ since $zy\in IJ$ $(z\in I, y\in J)$ and $\underbrace{zx = xz}_{\text{R is commutative}} \in IJ$ $(x\in I, z\in J)$.
Therefore, $I\cap J\subset IJ$.
Thus, we conclude that if $I+J=R$, then $IJ = I\cap J$. □

(b) Assuming that $I+R=J$, show that for any c and b in R there exists some $x\in R$ such that $x\equiv a \bmod I$ and $x\equiv b \bmod J$.
(Recall that $x\equiv a \bmod I$ if and only if $x-a\in I$.)

Pf: Since $I+J=R$, $\exists\, i\in I, j\in J$ s.t. we can write $i+j=1$.
Let $x = aj+bi\in R$.
Observe that (since $i+j=1$)

$$x\equiv aj+bi \equiv aj \bmod I \equiv aj+ai \bmod I \equiv a(i+j) \bmod I \equiv a \bmod I$$

$$x\equiv aj+bi \equiv bi \bmod J \equiv bi+bj \bmod J \equiv b(i+j) \bmod J \equiv b \bmod J$$

Therefore, $x\equiv a \bmod I$ and $x\equiv b \bmod J$.
Thus, we conclude that for any $a,b\in R$ $\exists\, x\in R$ s.t. $x\equiv a \bmod I$ and $x\equiv b \bmod J$. □

③ Let $\varphi : \mathbb{Z}\to \mathrm{Aut}(\mathbb{Z})$ by $n\mapsto \varphi_n$, where $\varphi_n(a) = (-1)^n a$. Define the semidirect product group $G = \mathbb{Z}\rtimes_\varphi \mathbb{Z}$.

(a) Write down the group law and the formula for inverses in G.

Pf: $(a,b)(x,y) = (a+\varphi_b(x), b+y) = (a+(-1)^b x, b+y)$

For inverse: let $(a+(-1)^b x, b+y) = (0,0)$

Then $a+(-1)^b x = 0$
$\Rightarrow (-1)^b x = -a$
$\Rightarrow x = \frac{-a}{(-1)^b} = \frac{a}{(-1)^{b-1}} = a(-1)^{1-b}$
and $b+y=0$
$\Rightarrow y=-b$
$\Rightarrow (a(-1)^{1-b}, -b)$

Check: $(a,b)(a(-1)^{1-b}, -b) = (a+\varphi_b(a(-1)^{1-b}), b-b)$
$= (a+(-1)^b a(-1)^{1-b}, 0)$
$= (a-a, 0)$
$= (0,0)$

The gp. law is $(a,b)(x,y) = (a+(-1)^b x, b+y)$ and the inverse of $(a,b)\in G$ is $(a(-1)^{1-b}, -b)$. □

(b) Find the center of G.

Pf: $Z(G) = \{x\in G : xg = gx\ \forall g\in G\} = \{x\in G : gxg^{-1} = x\ \forall g\in G\}$
Let $(a,b),(x,y)\in G$.
Assume $(x,y)\in Z(G)$

$(a,b)(x,y) = (a+(-1)^b x, b+y)$
$(x,y)(a,b) = (x+(-1)^y a, y+b)$
$\Big\}$ $x=a$, $y\equiv b \bmod 2$

(Needs to be cleaned up, a lot!)

$(a+(-1)^b x, b+y) = (x+(-1)^y a, y+b)$
We want to find x,y s.t. this holds $\forall a,b$

$a+(-1)^b x = x+(-1)^y a$
$(a,b)(x,y)(a,b)^{-1} = (a+(-1)^b x + a(-1)^{y+1}, y)$
Set $(x,y) = (a+(-1)^b x+a(-1)^{y+1}, y)$
$\Rightarrow x = a+(-1)^b x + a(-1)^{y+1} \Rightarrow x-(-1)^b x = a+a(-1)^{y+1}$
$x(1+(-1)^{b+1}) = a(1+(-1)^{y+1})$
if $a=0$, then $x(1+(-1)^{b+1}) = 0 \Rightarrow x=0$
If $(x,y)\in Z(G)$ then $(a=0, b=1)$ commutes w/ (x,y) $\Rightarrow 2x=0\Rightarrow x=0$
if y is even then $(0,y)\in Z(G)$.

$Z(G) = \{(0,2z) : z\in\mathbb{Z}\}$

④ In a commutative ring R, an ideal Q is called primary if whenever any a and b in R satisfy $ab\in Q$ and $a\notin Q$, we have $b^n\in Q$ for some integer $n\ge 1$. (Equivalently, if $ab\equiv 0 \bmod Q$ and $a\not\equiv 0 \bmod Q$, we have $b^n\equiv 0 \bmod Q$ for some integer $n\ge 1$. That is, in the ring R/Q any zero divisor is nilpotent.) Show that the nonzero primary ideals in a PID are the ideals of the form (p^n) where p is a prime element and n is a positive integer. You may use that a PID is a UFD.

Pf: We want to show that $P=(p^n)$, where P is a nonzero primary ideal in a PID R.
Since R is a PID, all nonzero primary ideals must be principal.
Let $P=(\alpha)$.
We WTS $\alpha = p^n$ for some p prime, $n\ge 1$.
Let q be a prime factor of α, $q\mid\alpha$.
Then we can write $\alpha = q\cdot\alpha'$, where $\alpha'\in R$.
$\alpha'\mid\alpha$, so $\alpha'\notin P \Rightarrow q^n\in P$, $n\ge 1$ by defn. of a primary ideal.
$q^n\in P\Rightarrow \alpha\mid q^n$.
Since PID $\Rightarrow$ UFD, the only factors of q^n are $1, q, q^2, \dots, q^n$.
Therefore, $\alpha = q^k$ for $1\le k\le n$.
$\alpha = (\text{unit}\cdot q^k) = (q^k)$. □

⑤ In $\mathbb{R}^3$ a line-plane pair is a pair of subspaces (V_1,V_2) where $V_1\subset V_2$, $\dim V_1 = 1$, and $\dim V_2 = 2$. The standard line-plane pair in $\mathbb{R}^3$ is $(\mathbb{R}e_1, \mathbb{R}e_1+\mathbb{R}e_2)$ where $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$. Let S be the set of all line-plane pairs in $\mathbb{R}^3$.

(a) The group $GL(3,\mathbb{R})$ of invertible 3×3 real matrices acts on S by $A\cdot(V_1,V_2) = (A(V_1), A(V_2))$, where $A\in GL(3,\mathbb{R})$ and $(V_1,V_2)\in S$. Prove that the stabilizer subgroup of the standard line-plane pair is the group of invertible upper-triangular matrices in $GL(3,\mathbb{R})$ (with arbitrary nonzero entries on the diagonal).

Pf: $A\cdot(\mathbb{R}e_1, \mathbb{R}e_1+\mathbb{R}e_2) = (A(\mathbb{R}e_1), A(\mathbb{R}e_1+\mathbb{R}e_2))$.
Let $A = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\in GL(3,\mathbb{R})$.

$$A(\mathbb{R}e_1) = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\mathbb{R} = \begin{pmatrix}a\\d\\g\end{pmatrix}\mathbb{R}$$

In order for this to be in the stabilizer, we need $\begin{pmatrix}a\\d\\g\end{pmatrix}\mathbb{R} = \begin{pmatrix}1\\0\\0\end{pmatrix}\mathbb{R}$
$\Rightarrow d=g=0$.

$$A(\mathbb{R}e_1+\mathbb{R}e_2) = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\mathbb{R} + \begin{pmatrix}0\\1\\0\end{pmatrix}\mathbb{R}\right) = \begin{pmatrix}a\\d\\g\end{pmatrix}\mathbb{R} + \begin{pmatrix}b\\e\\h\end{pmatrix}\mathbb{R}$$

In order for this to be in the stabilizer, we need $\begin{pmatrix}a\\d\\g\end{pmatrix}\mathbb{R} + \begin{pmatrix}b\\e\\h\end{pmatrix}\mathbb{R} = \begin{pmatrix}1\\0\\0\end{pmatrix}\mathbb{R} + \begin{pmatrix}0\\1\\0\end{pmatrix}\mathbb{R}$

$\Rightarrow d=g=0$ from before, and $b=h=0$.

Therefore, $A = \begin{pmatrix} a & 0 & c\\ 0 & e & f\\ 0 & 0 & i\end{pmatrix}\in GL(3,\mathbb{R})$, and A is upper-triangular.

The stabilizer subgp. of the standard line-plane pair is the group $\left\{\begin{pmatrix} a & 0 & c\\ 0 & e & f\\ 0 & 0 & i\end{pmatrix}\in GL(3,\mathbb{R})\right\}$ of invertible upper-triangular matrices in $GL(3,\mathbb{R})$ (w/ arbitrary non-zero entries on the diagonal.). □

(b) Prove that the $GL(3,\mathbb{R})$-action on S is transitive.

Pf: Transitive means only one orbit.
We want to show that the standard line-plane pair can be transformed to any line-plane pair.
Pick any plane and a vector in it (an arbitrary line-plane pair).
Pick $M\in GL(3,\mathbb{R})$.
Show that $M\cdot \underset{\text{LPP}}{\text{standard}} = \underset{\text{LPP}}{\text{arb. chosen}}$.

The standard LPP is in every orbit and the orbits are supposed to be either the same or disjoint.
Therefore, there is only one orbit. □

⑥ Give examples as requested, with brief justification.

(a) A maximal ideal in $\mathbb{C}[x,y]$ which contains the ideal (xy, x^2-1).

Pf: $(xy, x^2-1) = (xy, (x+1)(x-1))$
A maximal ideal in $\mathbb{C}[x,y]$ which contains the ideal (xy, x^2-1) is the ideal $(y, x+1)$.
It is clear that $(xy, x^2-1)\subset (y, x+1)$.
Observe that

$$\mathbb{C}[x,y]/(y,x+1) \cong \left(\mathbb{C}[x,y]/(y)\right)/(x+1) \cong \mathbb{C}[x]/(x+1) \cong \mathbb{C} = \text{field}.$$

Therefore, since $\mathbb{C}[x,y]/(y,x+1)$ is a field, we have that the ideal $(y, x+1)$ is maximal. □

(b) A ring R and ideals I and J in R such that $IJ\ne I\cap J$.

Pf: Let $R = 2\mathbb{Z}$, $I = 6\mathbb{Z}$ and $J = 8\mathbb{Z}$.
$I\cap J = 6\mathbb{Z}\cap 8\mathbb{Z} = 24\mathbb{Z}$
but $IJ = 6\mathbb{Z}\cdot 8\mathbb{Z} = 48\mathbb{Z}$
$IJ\ne I\cap J$ $(24\mathbb{Z}\ne 48\mathbb{Z})$. □