January 2021

martes, 20 de diciembre de 2022 12:00 p.m.

① a) If G is a group and $g,h \in G$ have orders m and n w/ $\gcd(m,n)=1$ and $gh=hg$, then prove gh has order mn.

Pf: $g^m = 1$ and $h^n = 1$ $\quad (m,n)=1$

So $(gh)^{mn} \underset{\downarrow}{=} g^{mn}h^{mn} = (g^m)^n (h^n)^m = 1^n 1^m = 1$

since $gh=hg$

$$g\underbrace{hg}_{\circlearrowright}hgh\cdots gh$$

$$gg\cdots hh\cdots$$

This tells us that $|gh| \mid mn$ $\quad (|gh| \leq mn)$

Note since $(m,n)=1$, $\operatorname{lcm}(m,n)=mn$

Suppose $(gh)^k = 1$ for $k \in \mathbb{Z}$

Then $(gh)^k = g^k h^k = 1$ (since $gh=hg$)

$\Rightarrow g^k = h^{-k}$

- Raising both sides to the mth power, we get

$\underset{\underset{(g^m)^k}{\downarrow}}{g^{km}} = h^{-km} \Rightarrow 1 = h^{-km} \Rightarrow h^{km}=1$

so $|h| \mid km$, but $(m,n)=1$

so $|h| \mid k$ $\qquad |h|=n$

$n \mid k$

- Raising both sides to the nth power, we get

$g^{kn} = h^{-kn} \Rightarrow g^{kn}=1$, so $|g| \mid kn$, but $(m,n)=1$

so $|g| \mid k$ $\qquad |g|=m$

$m \mid k$

Since $m \mid k$ and $n \mid k$ and $(m,n)=1 \Rightarrow mn \mid k$

$|gh| \leq mn$

Therefore, gh has order mn.

$\square$

b) The prime factorization of 2021 is $43 \cdot 47$. Use part (a) and the Sylow Theorems to prove that all groups of order 2021 are cyclic.

Pf: Let G be a finite group with $|G| = 2021 = 43 \cdot 47$

By the first Sylow Thm, there exists a 43-Sylow subgp. P of G w/ $|P| = 43$, and a 47-Sylow subgp. Q of G w/ $|Q| = 47$.

By the third Sylow Thm, we have

$$\left.\begin{array}{l} n_{43} \equiv 1 \mod 43, \ n_{43} \mid 47 \Rightarrow n_{43} = 1 \\ n_{47} \equiv 1 \mod 47, \ n_{47} \mid 43 \Rightarrow n_{47} = 1 \end{array}\right\} \begin{array}{l} P \text{ and } Q \text{ are the unique} \\ 43 \text{ and } 47\text{-Sylow subgps,} \\ \text{respectively.} \end{array}$$

$\Rightarrow P \lhd G$ and $Q \lhd G$

Since $|P| = 43$ and 43 is prime, we know that P is abelian

Since $|Q| = 47$ and 47 is prime, we know that Q is abelian

Since $P, Q \leq G$ and $P \lhd G$ (and $Q \lhd G$), we have that PQ is a subgp. of G.

Note that $|P \cap Q| = 1$: $P \cap Q \subset P$ and $P \cap Q \subset Q$, so by Lagrange's Thm, $|P \cap Q| \mid |P|$ and $|P \cap Q| \mid |Q|$, but $(|P|, |Q|) = (43, 47) = 1$, so $|P \cap Q| = 1$.

We have that $|PQ| = \dfrac{|P||Q|}{|P \cap Q|} = \dfrac{43 \cdot 47}{1} = 2021 = |G|$

Therefore, $G = PQ$.

We WTS that the elts. of P and Q commute: let $x \in P$, $y \in Q$

$$\underbrace{x}_{\in P}\underbrace{yx^{-1}y^{-1}}_{\substack{\in P \text{ since} \\ P \lhd G}} = \underbrace{xyx^{-1}}_{\substack{\in Q \\ \text{since} \\ Q \lhd G}}\underbrace{y^{-1}}_{\in Q} \in P \cap Q = \{1\} \Rightarrow xyx^{-1}y^{-1} = 1$$

$$xy = yx \ ✓$$

Therefore, the elts. of P, Q commute. Thus, G is abelian.

$$\left.\begin{array}{l} \text{Since } P \text{ is cyclic, it has an elt. } x \text{ of order } 43. \\ \text{Since } Q \text{ is cyclic, it has an elt. } y \text{ of order } 47. \end{array}\right\} (43, 47) = 1$$

By part (a), an elt. $xy \in G = PQ = \{pq : p \in P, q \in Q\}$ $(x \in P, y \in Q, |x| = 43, |y| = 47)$ has order $43 \cdot 47 = 2021$.

Therefore, G has an elt. of order $2021 = |G|$, so G is cyclic.

□

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② Let $G = GL_2(\mathbb{F}_3)$ act on the set X of all 1-dim. subspaces of $\mathbb{F}_3^2$:
$X = \left\{ \mathbb{F}_3\begin{pmatrix}1\\0\end{pmatrix}, \mathbb{F}_3\begin{pmatrix}0\\1\end{pmatrix}, \mathbb{F}_3\begin{pmatrix}1\\1\end{pmatrix}, \mathbb{F}_3\begin{pmatrix}2\\1\end{pmatrix} \right\}$ and for $A \in G$ and a 1-dim. subspace $L = \mathbb{F}_3 v$ in X, set $A(L) = \mathbb{F}_3 A(v)$.

(a) Prove this action of G on X has one orbit.

Pf: Let $\langle v \rangle = \mathbb{F}_3 v$

Notice that for v, w nonzero vectors, $A\langle v \rangle = \langle w \rangle$ iff $w \in A\langle v \rangle$

If $w \in A\langle v \rangle$, then $A\langle v \rangle$ is a 1-dim. subspace containing the nonzero vector w, so that is exactly $\langle w \rangle$.

Also note that $A\langle v \rangle = \langle Av \rangle$ by defn. of gp. action.

Now we compute:

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, so $\begin{pmatrix}0&1\\1&0\end{pmatrix}\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix}0\\1\end{pmatrix}\right\rangle$

$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, so $\begin{pmatrix}1&0\\1&1\end{pmatrix}\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$

$\begin{pmatrix}2&0\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$, so $\begin{pmatrix}2&0\\1&1\end{pmatrix}\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix}2\\1\end{pmatrix}\right\rangle$

These three lines show that the $\text{orb}_{\langle\binom{1}{0}\rangle}$ contains $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$.

Obviously the orbit contains $\begin{pmatrix}1\\0\end{pmatrix}$.

Therefore, we have proved that X is equal to the orbit of $\begin{pmatrix}1\\0\end{pmatrix}$.

□

(b) Compute the Stabilizer subgp. of $\mathbb{F}_3\begin{pmatrix}1\\0\end{pmatrix}$.

Pf: The Stabilizer of $\begin{pmatrix}1\\0\end{pmatrix}$ is exactly the matrices $A \in GL_2(\mathbb{F}_3)$ s.t. $\left\langle A\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$. This is equiv. to the condition $A\begin{pmatrix}1\\0\end{pmatrix} \in \left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$

Since $A\begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, we just need to find the matrices A s.t.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ or $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

Solving the first eqn. yields $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{matrix}a=1\\c=0\end{matrix}$

In order for A to be invertible, we must have $d \neq 0$. $(ad - \cancel{bc} \neq 0)$

So the first eqn. has soln. set $\left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b \in \mathbb{F}_3, d \in \mathbb{F}_3^{\times} \right\}$

Solving the second eqn. yields $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} a = 2 \\ c = 0 \end{matrix}$

In order for A to be invert., $d \neq 0$.

So the second eqn. has soln. set $\left\{ \begin{pmatrix} 2 & b \\ 0 & d \end{pmatrix} : b \in \mathbb{F}_3, d \in \mathbb{F}_3^{\times} \right\}$

The stabilizer is any matrix A that solves either of the first two eqns., so the stabilizer is exactly $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : b \in \mathbb{F}_3, a, d \in \mathbb{F}_3^{\times} \right\}$.

□

(c) Compute $|G|$.

Pf: By Orbit-Stabilizer Thm., we have $|\mathrm{Orb}_x| = [G : \mathrm{Stab}_x] = \dfrac{|G|}{|\mathrm{Stab}_x|}$

From (a), the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is X, $|X| = 4$

From (b), the Stab. of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has size $3 \cdot 2 \cdot 2 = 12$

Therefore, the order of G is $|\mathrm{Orb}_x||\mathrm{Stab}_x| = |G| = 48$.

$4 \cdot 12 = 48$

□

③ (a) Define what irreducibility means in an integral domain.

Pf: An elt. x in an integral domain R is irreducible if whenever we write $x = ab$, $a, b \in R$, then either a or b is a unit in R.

□

(b) Prove Eisenstein's irred. criterion for monic poly. in $\mathbb{Z}[x]$: if $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is monic in $\mathbb{Z}[x]$ and there is a prime p s.t. $p \mid a_j$ for $j < n$ and $p^2 \nmid a_0$, then $f(x)$ is irred. in $\mathbb{Z}[x]$.

Pf: Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ and let p be a prime s.t. $p \mid a_j$ for $j < n$ and $p^2 \nmid a_0$ (so $f(x)$ is Eisenstein)

Suppose $f(x)$ is reducible.

Then $f(x) = g(x)h(x)$ for $g(x), h(x) \in \mathbb{Z}[x]$, where $g(x), h(x)$ are nonconstant monic polys.

We have that $f(x) \equiv x^n \mod p$

$$f(0) \not\equiv 0 \mod p^2$$

So if we reduce $f(x) = g(x)h(x) \mod p$, we get

$x^n = \overline{g(x)}\,\overline{h(x)} \mod p$ b/c $p \mid a_j$ for $j < n$.

$x^n = \overline{g(x)h(x)}$ in $(\mathbb{Z}/p)[x]$

Note $\bar{g}$ is monic w/ $\deg(\bar{g}) = \deg(g) \geq 1$ and

$\bar{h}$ is monic w/ $\deg(\bar{h}) = \deg(h) \geq 1$

We know that $(\mathbb{Z}/p)[x]$ is a UFD ($(\mathbb{Z}/p)$ is a field), so by uniqueness of irred. factn., we get $\overline{g(x)} = x^i$ where $i \geq 1$

$\overline{h(x)} = x^{n-i}$ where $n-i \geq 1$

So $\overline{g(0)} = \bar{0} \Rightarrow g(0) \in p\mathbb{Z}$

$\overline{h(0)} = \bar{0} \Rightarrow h(0) \in p\mathbb{Z}$

Then $f(0) = g(0)h(0) \equiv 0 \mod p^2$ ↯ This contradicts $f(x)$ being Eisenstein

The constant term $f(0)$ should be div. by p just once.

Therefore, $f(x)$ is irred. in $\mathbb{Z}[x]$. □

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④ (a) In the ring $\mathbb{Z}[x]$, compute the index of the ideal $(2, x^2)$ in $\mathbb{Z}[x]$ and determine a maximal ideal that contains $(2, x^2)$.

Pf: The index of $(2, x^2)$ in $\mathbb{Z}[x]$ is equal to $\left|\mathbb{Z}[x]/(2, x^2)\right|$

We have that $\mathbb{Z}[x]/(2, x^2) \cong (\mathbb{Z}/(2))[x]/(x^2)$

$$\mathbb{Z}[x]/(2, x^2) \cong \left(\mathbb{Z}[x]/(2)\right)/(x^2) \cong (\mathbb{Z}/(2))[x]/(x^2)$$

We set $x^2 = 0$ in $(\mathbb{Z}/(2))[x]$

As a result, $(\mathbb{Z}/(2))[x]/(x^2) = \{a + bx : a, b \in \mathbb{Z}/(2)\}$, so the elts/rep.

of $(\mathbb{Z}/(2))[x]/(x^2)$ are $\{0, 1, x, 1+x\}$

Therefore, $\left|(\mathbb{Z}/(2))[x]/(x^2)\right| = 4$ and since $(\mathbb{Z}/(2))[x]/(x^2) \cong \mathbb{Z}[x]/(2,x^2)$

we have that $\left|\mathbb{Z}[x]/(2,x^2)\right| = 4.$

Thus, $(2,x^2)$ has index 4 in $\mathbb{Z}[x]$.

Now we want to determine a maximal ideal that contains $(2,x^2)$

- In $\mathbb{Z}[x]$, all max. ideals are of the form $(p, f(x))$ where p is a prime and $f(x)$ is a monic irred. poly. mod p.
- Therefore, $(2,x)$ is maximal in $\mathbb{Z}[x]$ since 2 is prime and x is irred. mod 2.

Consider $(2,x)$

Observe that $\mathbb{Z}[x]/(2,x) \cong (\mathbb{Z}/(2))[x]/(x) \cong \mathbb{Z}/2\mathbb{Z}$ = field

Therefore, $(2,x)$ is a max. ideal.

Observe that for $f(x) \in (2,x^2)$, we have that $f(x) = 2g(x) + x^2h(x)$
$(g(x), h(x) \in \mathbb{Z}[x])$
$$= 2g(x) + x(xh(x))$$
$$\in (2,x)$$

Therefore, $(2,x^2) \subset (2,x)$.

Thus, we conclude that $(2,x)$ is a max. ideal of $\mathbb{Z}[x]$ that contains $(2,x^2)$.

□

(b) In the ring $\mathbb{Z}[\sqrt{5}]$, show the principal ideal (3) is maximal.

Pf: Observe that $\mathbb{Z}[\sqrt{5}] \cong \mathbb{Z}[x]/(x^2-5)$

Then $$\mathbb{Z}[\sqrt{5}]/(3) \cong \left(\mathbb{Z}[x]/(x^2-5)\right)/(3) \cong \mathbb{Z}[x]/(3, x^2-5)$$

$$\cong \underbrace{(\mathbb{Z}/(3))}_{\text{field}}[x]/\underbrace{(x^2-5)}_{\text{irred. poly.}} \Big\} \text{ field}$$

The max. ideals of $\mathbb{Z}[x]$ are of the form $(\overset{\text{prime}}{p}, \overset{\text{irred. mod } p,\ \text{monic}}{f(x)})$

3 is prime

x^2-5 is irred. mod 3 $\longrightarrow$ $\left.\begin{array}{l} 0^2-5 \equiv 2\ (3) \\ 1^2-5 \\ 2^2-5 \end{array}\right\}$ can do this test b/c poly. is deg. 2

Therefore, $(3, x^2-5)$ is a maximal ideal of $\mathbb{Z}[x]$.

$$\underset{\text{field}}{\underset{\parallel}{\mathbb{Z}[x]/(3,x^2-5)}} \cong \left.\mathbb{Z}[x]/(x^2-5)\right/(3) \cong \mathbb{Z}[\sqrt{5}]/(3) = \text{field}$$

Therefore, (3) is maximal. $\square$

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⑤ On the ring $V = \mathbb{R}[x]/(x^2-4)$, viewed as a 2-dim. real vector space, there is an inner product defined by $\langle f, g\rangle = f(2)g(2) + f(-2)g(-2)$.

(a) For each $h \in V$, let $L_h : V \to V$ by $L_h(f) = hf$. Prove that L_h is self-adjoint for the inner product above.

Pf: If L_h is self-adjoint, then $\langle L_h(f), g\rangle = \langle f, L_h(g)\rangle$

Let $f, g \in V$. Observe that

$$\begin{aligned}
\langle L_h(f), g\rangle &= L_h(f)(2)\,g(2) + L_h(f)(-2)\,g(-2) \\
&= hf(2)\,g(2) + hf(-2)\,g(-2) \\
&= h(2)f(2)g(2) + h(-2)f(-2)g(-2) \\
&= f(2)h(2)g(2) + f(-2)h(-2)g(-2) \\
&= f(2)\,hg(2) + f(-2)\,hg(-2) \\
&= f(2)\,L_h(g)(2) + f(-2)\,L_h(g)(-2) \\
&= \langle f, L_h(g)\rangle
\end{aligned}$$

since V is a vector space $hf(x) = h(x)f(x)$

Therefore, $\langle L_h(f), g\rangle = \langle f, L_h(g)\rangle$, so we conclude that L_h is self-adjoint. $\square$

(b) Find the matrix repn. for L_{1+3x} w.r.t. the basis $\{1, x\}$ of V and use it to find a basis of eigenvectors in V for L_{1+3x}.

Pf: $L_{1+3x} : V \to V$ by $L_{1+3x}(f) = (1+3x)f$

First we want to find the matrix repn. for L_{1+3x}:

$$L_{1+3x}(1) = (1+3x)(1) = 1+3x = \underline{1}\cdot 1 + \underline{3}\cdot x$$

$$L_{1+3x}(x) = (1+3x)(x) = x+3x^2 = x + 3\cdot 4 = \underline{12}\cdot 1 + \underline{1}\cdot x$$
$$= x + 12$$

So the matrix repn. for L_{1+3x} wrt $\mathcal{B} = \{1, x\}$ of V is

$$[L_{1+3x}]_{\mathcal{B}} = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$$

Now we want to find the eigenvalues:

$$\det\left(\begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda\right) = \det\left(\begin{pmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2 - 36 = 1 - 2\lambda + \lambda^2 - 36$$
$$= \lambda^2 - 2\lambda - 35$$
$$= (\lambda - 7)(\lambda + 5)$$

The eigenvalues are 7 and -5.

Now we want to find the eigenvectors:

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$.

Then $\begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+12y \\ 3x+y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} x + 12y = 7x \\ 3x + y = 7y \end{matrix}$

$$\left.\begin{matrix} \Rightarrow 12y = 6x \Rightarrow 2y = x \\ \Rightarrow 3x = 6y \Rightarrow x = 2y \end{matrix}\right\} 2y = x \Rightarrow y = \tfrac{1}{2}x$$

Therefore, the eigenvectors of 7 are of the form $\begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$

Then $\begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+12y \\ 3x+y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} x + 12y = -5x \\ 3x + y = -5y \end{matrix}$

$$\left.\begin{matrix} \Rightarrow 12y = -6x \\ \Rightarrow 3x = -6y \end{matrix}\right\} 2y = -x \Rightarrow y = \tfrac{-1}{2}x$$

Therefore, the eigenvectors of -5 are of the form $\begin{pmatrix} x \\ -\frac{1}{2}x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$.

Thus, the basis of eigenvectors is $\left\{\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\right\}$.

□

⑥ Give examples as requested, with justification.

(a) A group isomorphism $A_4 \rtimes_\varphi \mathbb{Z}/(2) \to S_4$ for some φ.

Pf: We have that $S_4 \cong A_4 \rtimes_\varphi \mathbb{Z}/(2)$ where $\mathbb{Z}/(2)$ is identified with $\{1, \tau\}$ for any transposition τ.
Let $\varphi: \mathbb{Z}/(2) \to \mathrm{Aut}(A_4)$

$$k \longmapsto \sigma \mapsto \tau^k \sigma \tau^{-k}$$

□

(b) An irred. factn. of 13 in $\mathbb{Z}[\sqrt{3}]$.

Pf: $13 = (5 + 2\sqrt{3})(5 - 2\sqrt{3})$.
We have that $N(5+2\sqrt{3}) = (5+2\sqrt{3})(5-2\sqrt{3}) = 13$
Let $5 \pm 2\sqrt{3} = \alpha\beta$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{3}]$
Then $N(5 \pm 2\sqrt{3}) = 13 = N(\alpha)N(\beta)$, so $N(\alpha) \mid 13 \Rightarrow N(\alpha) = \pm 1, \pm 13$
Therefore, either α or β is a unit. Thus, $5 \pm 2\sqrt{3}$ is irred.
Thus, $13 = (5+2\sqrt{3})(5-2\sqrt{3})$ is an irred. factn. of 13 in $\mathbb{Z}[\sqrt{3}]$.

□

(c) A ring isomorphism $\mathbb{R}[x]/(x^2+3) \cong \mathbb{C}$.

Pf: Let $\varphi: \mathbb{R}[x] \to \mathbb{C}$ by $f(x) \mapsto f(\sqrt{-3})$
φ is the evaluation map which is a hom. and onto.
We WTS: $\ker(\varphi) = (x^2+3)$.

- if $f(x) \in (x^2+3)$, then $f(x) = (x^2+3)g(x) \Rightarrow \varphi(f(x)) = 0$
$\Rightarrow f(x) \in \ker(\varphi)$.
So $(x^2+3) \subseteq \ker(\varphi)$.
- $f(x) = (x - \sqrt{-3})(x + \sqrt{-3})h(x) = (x^2+3)h(x) \in (x^2+3)$
so $\ker(\varphi) \subseteq (x^2+3)$.

Therefore, $\ker(\varphi) = (x^2+3)$.
Thus, by the first isom. thm., $\mathbb{R}[x]/(x^2+3) \cong \mathbb{C}$.

□

(d) A nonzero elt. in the dual space $(\mathbb{R}^2)^*$ that is in the kernel of the dual to the linear map $\begin{pmatrix}1&3\\2&6\end{pmatrix}: \mathbb{R}^2 \to \mathbb{R}^2$. (Write an answer as a linear combination of the dual basis of the standard basis of $\mathbb{R}^2$.)

Pf: · Let $L = \begin{pmatrix}1&3\\2&6\end{pmatrix}$ be viewed as a linear map $L: \mathbb{R}^2 \to \mathbb{R}^2$, i.e.,
$L(v) = \begin{pmatrix}1&3\\2&6\end{pmatrix}v$ for all $v \in \mathbb{R}^2$.

- Then the dual map to L, call it L^*, is the linear map from $(\mathbb{R}^2)^*$ to itself given by $L^*(\varphi) = \varphi \circ L$ where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a linear functional.

- We want to find a nonzero φ that belongs to the kernel of L^*, so we need to find φ satisfying $L^*(\varphi) = 0$ where the RHS denotes the zero map from $\mathbb{R}^2$ to $\mathbb{R}$.

- Since $L^*(\varphi)$ is linear, it makes sense to see how L^* acts on the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$:

$$L^*(\varphi)(e_1) = (\varphi \circ L)(e_1) = \varphi\left(\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \varphi\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \varphi(e_1) + 2\varphi(e_2).$$

$$L^*(\varphi)(e_2) = (\varphi \circ L)(e_2) = \varphi\left(\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \varphi\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\varphi(e_1) + 6\varphi(e_2).$$

- Notice that $L^*(\varphi)$ is the zero map iff $L^*(\varphi)(e_1) = L^*(\varphi)(e_2) = 0$.

  From the above two equations, we see that if $\varphi(e_1) = -2\varphi(e_2)$, then $L^*(\varphi)(e_i) = 0$ for $i = 1, 2$.

- With this in mind, it suffices to find a functional $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which satisfies $\varphi(e_1) = 2$ and $\varphi(e_2) = -1$.

  For instance, we could choose $\varphi := 2e_1^* - e_2^*$ where $\{e_1^*, e_2^*\}$ is the dual basis to the standard basis of $\mathbb{R}^2$.

$\square$