continued...

(4) Prove every $\mathbb{R}$-linear ~~transformed~~ transformation $L: \mathbb{C} \to \mathbb{C}$ can be described by the rule $L(z) = \alpha z + \beta \bar{z}$ for a unique pair of complex numbers α and β by constructing an $\mathbb{R}$-linear isomorphism $\mathbb{C}^2 \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.

→ Let $f: \mathbb{C}^2 \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ by $f(\alpha,\beta) = \phi_{\alpha,\beta}$, where $\phi_{\alpha,\beta}: \mathbb{C} \to \mathbb{C}$ such that $\phi_{\alpha,\beta}(z) = \alpha z + \beta \bar{z}$.

First, we want to show that $\phi_{\alpha,\beta}$ is an $\mathbb{R}$-linear transformation from $\mathbb{C}$ to $\mathbb{C}$:

$\phi_{\alpha,\beta}(z_1 + z_2) = \alpha(z_1 + z_2) + \beta(\overline{z_1 + z_2}) = \alpha(z_1+z_2) + \beta(\bar{z}_1 + \bar{z}_2) = \alpha z_1 + \beta \bar{z}_1 + \alpha z_2 + \beta \bar{z}_2$

$= \phi_{\alpha,\beta}(z_1) + \phi_{\alpha,\beta}(z_2)$.

Also for $c \in \mathbb{R}$, $\phi_{\alpha,\beta}(cz) = \alpha c z + \beta(\overline{cz}) = c\alpha z + \underbrace{\beta c \bar{z}}_{\text{since } c \in \mathbb{R}} = c(\alpha z + \beta \bar{z}) = c\phi_{\alpha,\beta}(z)$.

Thus $\phi_{\alpha,\beta}$ is $\mathbb{R}$-linear for any $\alpha, \beta \in \mathbb{C}$.

Now, I claim that f is an $\mathbb{R}$-linear isomorphism. Notice that

$f((\alpha,\beta) + (\alpha',\beta')) = f((\alpha+\alpha', \beta+\beta')) = \phi_{\alpha+\alpha',\beta+\beta'}$ and that

$f(\alpha,\beta) + f(\alpha',\beta') = \phi_{\alpha,\beta} + \phi_{\alpha',\beta'}$. But, $\phi_{\alpha,\beta}(z) + \phi_{\alpha',\beta'}(z) = \alpha z + \beta \bar{z} + \alpha' z + \beta' \bar{z}$

$= (\alpha + \alpha')z + (\beta+\beta')\bar{z} = \phi_{\alpha+\alpha',\beta+\beta'}(z)$.

Also for $c \in \mathbb{R}$, $f(c(\alpha,\beta)) = f(c\alpha, c\beta) = \phi_{c\alpha,c\beta}$ where $\phi_{c\alpha,c\beta}(z) = c\alpha z + c\beta \bar{z} = c(\alpha z + \beta \bar{z})$

$= c\phi_{\alpha,\beta}(z)$ and $c\phi_{\alpha,\beta} = cf(\alpha,\beta)$.

Thus, f is an $\mathbb{R}$-linear transformation.

To show that f is injective, computing the kernel, we see that

$\ker(f) = \{(\alpha,\beta) \in \mathbb{C}^2 : \phi_{\alpha,\beta}(z) = 0 \ \forall z \in \mathbb{C}\}$.

- If $\phi_{\alpha,\beta}$ is the zero map, then $\phi_{\alpha,\beta}(z) = \alpha z + \beta \bar{z} = 0 \ \forall z \in \mathbb{Z}$. Let $z = 1$. Then $\phi_{\alpha,\beta}(1) = 0$ $\Leftrightarrow \alpha = -\beta$. Similarly, letting $z = i$ we see that $\phi_{\alpha,\beta}(i) = 0 \Leftrightarrow \alpha i - \beta i = 0 \Leftrightarrow \alpha i = \beta i$ $\Leftrightarrow \alpha = \beta$. Thus, we must have $\alpha = \beta$ and $\alpha = -\beta$, hence $\alpha = \beta = 0$. Therefore, $\ker(f) = \{(0,0)\}$ which shows that f is injective.
- For surjectivity, we know that if $L: V \to \tilde{V}$ is a linear transformation and $\dim(V) = \dim(\tilde{V}) < \infty$, then L is inj. iff it is surj. Thus, we must only show that $\dim(\mathbb{C}^2) = \dim(\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}))$. Since $\dim(\mathbb{C}^2) = \dim(\mathbb{C}) + \dim(\mathbb{C})$, we see $\dim(\mathbb{C}^2) = 4$ (it can also be seen that since $\{1, i\}$ is a basis for $\mathbb{C}$, $\{(1,0),(i,0),(0,1),(0,i)\}$ is a basis for $\mathbb{C}^2$). We know that for vector spaces V and W over F, the dimension of $\mathrm{Hom}_F(V,W) = (\dim V)(\dim W)$. Thus, we have $\dim_{\mathbb{R}}(\mathbb{C},\mathbb{C}) = (\dim \mathbb{C})(\dim \mathbb{C}) = 4$. Hence, $\dim(\mathbb{C}^2) = \dim(\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})) = 4$, which shows that f is surjective.

Therefore, f is an $\mathbb{R}$-linear isomorphism.

By the surj. of f, we know that every $\mathbb{R}$-linear transformation ϕ from $\mathbb{C}$ to $\mathbb{C}$ is of the form $\phi(z) = \alpha z + \beta \bar{z}$, and by the inj. of f, we know that α and β are unique.

~~HW Set 6~~

① In this exercise, treat $\mathbb{C}$ as a real vector space.

a) For $w \in \mathbb{C}$, let $\varphi_w : \mathbb{C} \to \mathbb{R}$ by $\varphi_w(z) = \mathrm{Re}(zw)$, where Re means the real part: $\mathrm{Re}(a+bi) = a$ for real a and b. Show φ_w is $\mathbb{R}$-linear, show $w \mapsto \varphi_w$ is an $\mathbb{R}$-linear isomorphism $\mathbb{C} \to \mathbb{C}^\vee$, and find $w \in \mathbb{C}$ such that $\varphi_w = \mathrm{Im}$ (the imaginary part function on $\mathbb{C}$).

→ First, we will show that φ_w is $\mathbb{R}$-linear. We have that $\varphi_w(z) = \mathrm{Re}(zw)$, and $\varphi_w(z_1 + z_2) = \mathrm{Re}((z_1+z_2)w) = \mathrm{Re}(z_1w + z_2w) = \mathrm{Re}(z_1w) + \mathrm{Re}(z_2w) = \varphi_w(z_1) + \varphi_w(z_2)$, and $\varphi_w(cz) = \mathrm{Re}(czw) = c\,\mathrm{Re}(zw) = c\varphi_w(z)$, for $c \in \mathbb{R}$. Therefore, φ_w is $\mathbb{R}$-linear.

→ Second we will show that $w \mapsto \varphi_w$ is an $\mathbb{R}$-linear isomorphism $\mathbb{C} \to \mathbb{C}^\vee$. Define $L : \mathbb{C} \to \mathbb{C}^\vee$ by $L(w) = \varphi_w$. First, we WTS that L is $\mathbb{R}$-linear: $L(w_1 + w_2) = \varphi_{w_1 + w_2} = \varphi_{w_1} + \varphi_{w_2} = L(w_1) + L(w_2)$. It suffices to show that $\varphi_{w_1+w_2}(z) = \varphi_{w_1}(z) + \varphi_{w_2}(z)$: $\varphi_{w_1+w_2}(z) = \mathrm{Re}(z(w_1+w_2)) = \mathrm{Re}(zw_1 + zw_2) = \mathrm{Re}(zw_1) + \mathrm{Re}(zw_2) = \varphi_{w_1}(z) + \varphi_{w_2}(z)$, and for $\lambda L(w) = \lambda\varphi_w$ it suffices to show $\lambda\varphi_w = \varphi_{\lambda w}$: $\lambda\varphi_w(z) = \lambda\mathrm{Re}(zw) = \mathrm{Re}(z\lambda w) = \varphi_{\lambda w}(z)$ for $\lambda \in \mathbb{C}$. Therefore, we have that L is $\mathbb{R}$-linear.

→ Now we WTS that L is bijective. Since $\mathbb{C}, \mathbb{C}^\vee$ have the same dimension, it suffices to show injectivity or surjectivity in order to show that L is bijective. We can show injectivity by showing the kernel is trivial as follows. We have that $\ker(L) = \{w \in \mathbb{C} : \varphi_w \equiv 0\}$ $= \{w \in \mathbb{C} : \varphi_w(z) = \mathrm{Re}(zw) = 0 \ \forall z \in W\}$. Let $w = a+bi$. If $z = 1$, then $a = 0$. If $z = -i$, then $b = 0$. Therefore, $w = 0$ and we conclude that $\ker(L) = \{0\}$.

→ Lastly, we want to find $w \in \mathbb{C}$ s.t. $\varphi_w = \mathrm{Im}$. In other words, we want to find $w \in \mathbb{C}$ s.t. $\varphi_w(z) = \mathrm{Re}(zw) = \mathrm{Im}(z)$. Let $(a+bi)w = b + ai$, where $a + bi = z$. Then $(a+bi)(-i) = b - ai$. So let $w = -i$, then $\varphi_{-i}(z) = \mathrm{Re}(z(-i)) = \mathrm{Re}(-zi) = \mathrm{Im}(z)$. Thus, $w = -i$.

b) Let $L, M : \mathbb{C} \to \mathbb{C}$ by $L(z) = \bar{z}$ and $M(z) = (5+3i)z$ for all $z \in \mathbb{C}$. Both are $\mathbb{R}$-linear, so they have dual linear maps $L^\vee, M^\vee : \mathbb{C}^\vee \to \mathbb{C}^\vee$. For each $w = a+bi$ in $\mathbb{C}$, write both $L^\vee(\varphi_w) = \varphi_w \circ L$ and $M^\vee(\varphi_w) = \varphi_w \circ M$ as explicit $\mathbb{R}$-linear combinations of φ_i and φ_{2-i} where the coefficients are in terms of a and b. Use this to determine the matrix representations of both $L^\vee$ and $M^\vee$ with respect to the basis $\{\varphi_i, \varphi_{2-i}\}$ of $\mathbb{C}^\vee$.

→ First we want to write $L^\vee(\varphi_w)$ and $M^\vee(\varphi_w)$ as explicit $\mathbb{R}$-linear combinations of φ_i and φ_{2-i}. For $L^\vee(\varphi_w)$ we have: $L^\vee(\varphi_w(z)) = (\varphi_w \circ L)(z) = \varphi_w(L(z)) = \varphi_w(\bar{z}) = \mathrm{Re}(\bar{z}w) = \mathrm{Re}(\bar{z}(a+bi))$ $= \mathrm{Re}(\bar{z}a + \bar{z}bi) = \mathrm{Re}(\bar{z}a) + \mathrm{Re}(\bar{z}bi) = a\mathrm{Re}(z) + b\mathrm{Im}(z)$. So, $L^\vee(\varphi_w(z)) = a\mathrm{Re}(z) + b\mathrm{Im}(z)$. For $M^\vee(\varphi_w)$ we have: $M^\vee(\varphi_w(z)) = (\varphi_w \circ M)(z) = \varphi_w(M(z)) = \varphi_w((5+3i)z) = \mathrm{Re}((5+3i)zw)$ $= \mathrm{Re}((5+3i)(a+bi)w) = \mathrm{Re}((5a + 5bi + 3ai - 3b)z) = \mathrm{Re}((5a-3b)z + (5b+3a)zi)$ $= \mathrm{Re}((5a-3b)z) + \mathrm{Re}((5b+3a)zi) = (5a-3b)\mathrm{Re}(z) - (5b+3a)\mathrm{Im}(z)$. So, we have that $M^\vee(\varphi_w(z)) = (5a-3b)\mathrm{Re}(z) - (5b+3a)\mathrm{Im}(z)$.

· Then we want to compute φ_i and φ_{2-i}: $\varphi_i(z) = \mathrm{Re}(iz) = -\mathrm{Im}(z)$ and $\varphi_{2-i}(z) = \mathrm{Re}((2-i)z) = \mathrm{Re}(2z) - \mathrm{Re}(iz) = 2\mathrm{Re}(z) - \mathrm{Im}(\bar{z}) = 2\mathrm{Re}(z) + \mathrm{Im}(z)$. So $\varphi_i(z) = -\mathrm{Im}(z)$ and $\varphi_{2-i}(z) = 2\mathrm{Re}(z) + \mathrm{Im}(z)$. Note that $\frac{1}{2}\varphi_i(z) + \frac{1}{2}\varphi_{2-i}(z) = \mathrm{Re}(z)$. Now, we can express $L^\vee(\varphi_w)$ and $M^\vee(\varphi_w)$ in terms of φ_i and φ_{2-i}: $L^\vee(\varphi_w(z)) = a\mathrm{Re}(z) + b\mathrm{Im}(z) = a\frac{1}{2}(\varphi_i(z) + \varphi_{2-i}(z)) - b\varphi_i(z)$ $= \frac{a}{2}\varphi_{2-i}(z) + (\frac{a}{2} - b)\varphi_i(z)$. And $M^\vee(\varphi_w(z)) = (5a-3b)\mathrm{Re}(z) - (5b+3a)\mathrm{Im}(z) = \frac{(5a-3b)}{2}(\varphi_{2-i}(z) + \varphi_i(z))$ $+ (5b+3a)\varphi_i(z) = \frac{5a-3b}{2}\varphi_{2-i}(z) + \left(\frac{5a-3b}{2} + (5b+3a)\right)\varphi_i(z) = \frac{5a-3b}{2}\varphi_{2-i}(z) + \frac{11a+7b}{2}\varphi_i(z)$.

So, $L^\vee(\varphi_w(z)) = (\frac{a}{2} - b)\varphi_i(z) + \frac{a}{2}\varphi_{2-i}(z)$ and $M^\vee(\varphi_w(z)) = \left(\frac{11a+7b}{2}\right)\varphi_i(z) + \left(\frac{5a-3b}{2}\right)\varphi_{2-i}(z)$. Now we will use this to determine the matrix rep of both $L^\vee$ and $M^\vee$ with respect to the basis $\mathcal{B} = \{\varphi_i, \varphi_{2-i}\}$ of $\mathbb{C}^\vee$. (Note that for $\varphi_i \Rightarrow a = 0, b = 1$ and for $\varphi_{2-i} \Rightarrow a = 2, b = -1$): $[L^\vee]_{\mathcal{B}} = ([L^\vee(\varphi_i)]_{\mathcal{B}} \ [L^\vee(\varphi_{2-i})]_{\mathcal{B}}) = ([\mathrm{Im}]_{\mathcal{B}} \ [2\mathrm{Re} - \mathrm{Im}]_{\mathcal{B}})$ $= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, and $[M^\vee]_{\mathcal{B}} = ([M^\vee(\varphi_i)]_{\mathcal{B}} \ [M^\vee(\varphi_{2-i})]_{\mathcal{B}}) = ([-3\mathrm{Re} - 5\mathrm{Im}]_{\mathcal{B}} \ [13\mathrm{Re} - \mathrm{Im}]_{\mathcal{B}}) = \begin{pmatrix} 7/2 & 15/2 \\ -3/2 & 13/2 \end{pmatrix}$.

continued...

② For $n \geq 1$, let $Pol_n(\mathbb{R})$ be the space of real polynomials with degree at most n: $Pol_n(\mathbb{R}) = \{a_0 + a_1X + \ldots + a_nX^n\}$. We are going to realize its $\mathbb{R}$-dual space as differential operators evaluated at 0. Set $D = \frac{d}{dx}$ to be differentiation, D^j to be the j^{th} derivative, and let $D_n(\mathbb{R})$ be the space of constant coefficient linear differential operators of order at most n: $D_n(\mathbb{R}) = \{b_0 + b_1D + \ldots + b_nD^n : b_i \in \mathbb{R}\}$. For instance, $(2 - D + D^2)(f) = 2f(x) - f'(x) + f''(x)$. (Note the constant b_0 in $D_n(\mathbb{R})$ is the operator "multiply by b_0"). We write a typical element of $Pol_n(\mathbb{R})$ as f or $f(X)$ and a typical element of $D_n(\mathbb{R})$ as L (for "linear").

a) When $f = a_0 + a_1X + \ldots + a_nX^n$ and $L = b_0 + b_1d + \ldots + b_nD^n$, compute $(Lf)(0)$ in terms of the coefficients of f and L.

→ We can express L in terms of f as follows: $L(f) = b_0f + b_1D(f) + b_2D^2(f) + \ldots + b_nD^n(f)$ $= b_0f + b_1f^{(1)} + b_2f^{(2)} + \ldots + b_nf^{(n)}$. Now we want to find a way to express $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$. We know that $f^{(1)} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} = \sum_{i=1}^{n} a_ix^{i-1}(i)$. Let $f(x) = \sum_{i=0}^{n} a_ix^i$, then

$f^{(1)}(x) = \sum_{i=1}^{n} ia_ix^{i-1}$, when $i-1 \geq 1$ it is 0, so $1a_1 = a_1$,

$f^{(2)}(x) = \sum_{i=1}^{n} i(i-1)a_ix^{i-2}$, when $i-2 \geq 1$ it's 0, so $2(1)a_2 = 2a_2$, ...

$f^{(j)}(x) = \sum_{i=j}^{n} (i)(i-1)\ldots(i-j+1)a_ix^{i-j}$, when $i-j \geq 1$ it's 0, so $j(j-1)\ldots(1)a_j = j!a_j = f^{(j)}(0)$.

Now we can express $(L(f))(0)$ as follows: $(L(f))(0) = b_0f(0) + b_1f^{(1)}(0) + b_2f^{(2)}(0) + \ldots + b_nf^{(n)}(0)$ $= b_0a_0 + b_1a_1 + b_2(2a_2) + b_3(3!a_3) + \ldots + b_k(k!a_k) + \ldots + b_n(n!a_n) = \sum_{i=0}^{n} i!b_ia_i$.

So $(Lf)(0) = \sum_{i=0}^{n} i!a_ib_i$.

b) For $L \in D_n(\mathbb{R})$, define $L_0 : Pol_n(\mathbb{R}) \to \mathbb{R}$ by $L_0(f) = (Lf)(0)$. Show L_0 is $\mathbb{R}$-linear and the mapping $L \mapsto L_0$ is an $\mathbb{R}$-vector space isomorphism $D_n(\mathbb{R}) \to Pol_n(\mathbb{R})^\vee$. (Start by showing $1, D, \ldots, D^n$ are linearly independent operators $\mathbb{R}[X] \to \mathbb{R}[X]$, so $\dim D_n(\mathbb{R}) = n+1$).

→ We will first show that $L_0 : Pol_n(\mathbb{R}) \to \mathbb{R}$ by $L_0(f) = (Lf)(0)$ is $\mathbb{R}$-linear. Let $f, g \in Pol_n(\mathbb{R})$ s.t. $f(x) = \sum_{k=0}^{n} a_kx^k$ and $g(x) = \sum_{k=0}^{n} c_kx^k$. Then $L_0(f+g) = (L(f+g))(0) = L\left(\sum_{k=0}^{n}(a_k + c_k)x^k\right)(0)$ $= \sum_{k=0}^{n} k!(a_k + c_k)b_k = \sum k!b_ka_k + \sum k!b_kc_k = (Lf)(0) + (Lg)(0) = L_0(f) + L_0(g)$. Let $d \in \mathbb{R}$. Then $L_0(df) = (L(df))(0) = L\left(\sum_{k=0}^{n} da_kx^k\right)(0) = \sum_{k=0}^{n} k!db_ka_k = d\sum k!b_ka_k = d(L(f))(0) = dL_0(f)$.

Thus, L_0 is $\mathbb{R}$-linear.

- Now we claim that $\varphi : D_n(\mathbb{R}) \to (Pol_n(\mathbb{R}))^\vee$ by $\varphi(L) = L_0$ is an isomorphism. First, notice that φ is $\mathbb{R}$-linear since for $L, L' \in D_n(\mathbb{R})$ s.t. $L = b_0 + b_1D + \ldots + b_nD^n$ and $L' = b_0' + b_1'D + \ldots + b_n'D^n$, $\varphi(L + L') = (L + L')_0$ and $\varphi(L) + \varphi(L') = L_0 + L_0'$ where $L + L' = (b_0 + b_0') + (b_1 + b_1')D + \ldots + (b_n + b_n')D^n$ so for $f \in Pol_n(\mathbb{R})$ s.t. $f = a_0 + a_1x + \ldots + a_nx^n$, then $(L+L')_0(f) = ((L+L')f)(0) = \sum_{k=0}^{n} k!a_k(b_k + b_k')$ and $L_0(f) + L_0'(f) = \sum_{k=0}^{n} k!a_kb_k + \sum_{k=0}^{n} k!a_kb_k'$ $= \sum k!a_k(b_k + b_k')$. Hence, $\varphi(L + L') = \varphi(L) + \varphi(L')$. Also for $d \in \mathbb{R}$, $\varphi(dL) = (dL)_0$ where $(dL)_0(f) = \sum_{k=0}^{n} k!a_kdb_k = d\sum k!a_kb_k = d(L_0(f))$. Hence, $\varphi(dL) = d\varphi(L)$. Thus, φ is $\mathbb{R}$-linear.

- Computing the kernel we see $\ker(\varphi) = \{L \in D_n(\mathbb{R}) : \varphi(L) = 0\} = \{L : (Lf)(0) = 0 \ \forall f \in Pol_n(\mathbb{R})\}$. But $(Lf)(0) = 0$ iff for $L = b_0 + b_1D + \ldots + b_nD^n$ and for any $f(x) = a_0 + a_1X + \ldots + a_nX^n$, we have $\sum_{k=0}^{n} k!a_kb_k = 0$. Then for $\ell = 0, 1, 2, \ldots, n$, let $f_\ell(x) = x^\ell$. Then $(Lf_\ell)(0) = \sum_{k=0}^{n} k!a_kb_k$ $= b_\ell = 0$ for all ℓ. Hence, we must have $L = 0$ so that $\ker(\varphi) = \{0\}$ showing that φ is inj..

Now we WTS φ is surj.. It suffices to show $\dim D_n(\mathbb{R}) = n+1$ since we know that $\{1, x, \ldots, x^n\}$ is a basis for $Pol_n(\mathbb{R})$, and thus $Pol_n(\mathbb{R})$ and $Pol_n(\mathbb{R})^\vee$ have dim $= n+1$. We claim $\{1, D, \ldots, D^n\}$ is lin. ind. Suppose that $L = b_0 + b_1D + \ldots + b_nD^n = 0$, $b_j \in \mathbb{R}$. Then $Lf = 0 \ \forall f \in Pol_n(\mathbb{R})$. Let $f(x) = x^n$. Then $Lf = b_0f + b_1f' + \ldots + b_nf^n = b_0x^n + b_1nx^{n-1} + b_2n(n-1)x^{n-2} + \ldots + b_{n-1}n!x + b_nn! = 0$. Since $\{1, x, \ldots, x^n\}$ is a basis for $Pol_n(\mathbb{R})$, it is a lin. ind set, thus, we must have that $b_0 = b_1n = b_2n(n-1) = \ldots = b_{n-1}n! = b_nn! = 0$. Hence $b_i = 0 \ \forall i = 0, 1, \ldots, n$. Thus, $\{1, D, \ldots, D^n\}$ is a lin. ind. set that spans $D_n(\mathbb{R})$. Hence it is a basis of $D_n(\mathbb{R})$ so that $D_n(\mathbb{R})$ has dim $= n+1$. Therefore, φ is an isomorphism.

c) Viewing $D_n(\mathbb{R})$ as $\mathrm{Pol}_n(\mathbb{R})^\vee$ using the isomorphism in part (b), determine the dual basis in $D_n(\mathbb{R})$ to the basis $\{1, X, X^2, \dots, X^n\}$ of $\mathrm{Pol}_n(\mathbb{R})$.

→ Recall that $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathrm{Pol}_n(\mathbb{R})$ so that $\{1^\vee, x^\vee, \dots, (x^n)^\vee\}$ is the dual basis in $(\mathrm{Pol}_n(\mathbb{R}))^\vee$ where for $k = 0, 1, \dots, n$, $(x^k)^\vee : \mathrm{Pol}_n(\mathbb{R}) \to \mathbb{R}$ by $(x^k)^\vee(a_0 + a_1x + \dots + a_nx^n) = a_k$, i.e., $(x^k)^\vee$ projects the coefficient of x^k in f. Notice that $1^\vee(f) = a_0 = f(0)$, $x^\vee(f) = a_1 = f'(0)$, $(x^2)^\vee(f) = a_2 = \frac{1}{2}f''(0), \dots, (x^n)(f) = a_n = \frac{1}{n!}f^n(0)$. Let D_0^n denote $(Lf)(0)$ for $L = D^n$. By the isomorphism from part (b), we see that $1^\vee = 1_0 \xrightarrow{\varphi^{-1}} 1$, $x^\vee = D_0 \xrightarrow{\varphi^{-1}} D$, $(x^2)^\vee = \frac{1}{2}D_0^2 \xrightarrow{\varphi^{-1}} \frac{1}{2}D^2, \dots,$ $(x^n)^\vee = \frac{1}{n!}D_0^n \xrightarrow{\varphi^{-1}} \frac{1}{n!}D^n$. Thus, the dual basis in $D_n(\mathbb{R})$ is $\{1, D, \frac{1}{2}D, \dots, \frac{1}{n!}D^n\}$.

d) Two linear maps $\mathrm{Pol}_n(\mathbb{R}) \to \mathbb{R}$ are $f(x) \mapsto f(1)$ and $f(x) \mapsto \int_0^1 f(x)\,dx$. According to part (a) there are differential operators $L, M \in D_n(\mathbb{R})$ such that $f(1) = (Lf)(0)$ and $\int_0^1 f(x)\,dx = (Mf)(0)$ for all $f \in \mathrm{Pol}_n(\mathbb{R})$. What are L and M?

→ First we will solve for L. Let $f(x) = a_0 + a_1x + \dots + a_nx^n$ and $L = b_0 + b_1D + \dots + b_nD^n$. We know that $(Lf)(0) = \sum_{i=1}^n i!\,b_ia_i = f(1) = \sum_{i=1}^n a_i$, so $b_i = \frac{1}{i!}$. Therefore, $L = 1 + \frac{1}{1!}D + \frac{1}{2!}D^2 + \dots + \frac{1}{n!}D^n$.

Now we will solve for M. We have that $(Mf)(0) = \int_0^1 f(x)\,dx = \int_0^1 (a_0 + a_1x + \dots + a_nx^n)\,dx$
$= \left(a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots + \frac{a_n}{n+1}x^{n+1}\right)\Big|_0^1 = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = \sum_{i=1}^n \frac{a_i}{i+1} = \sum_{i=0}^n i!\,b_ia_i = (Mf)(0)$.
Therefore, $M = 1 + \frac{1}{2!}D + \frac{1}{3!}D^2 + \dots + \frac{1}{(n+1)!}D^n$.

③ Let F be a field. This exercise show all elements of $M_n(F)^\vee$ can be built from the trace map $\mathrm{Tr}: M_n(F) \to F$ where $\mathrm{Tr}((a_{ij})) = \sum_{i=1}^n a_{ii}$, a specific linear functional on $M_n(F)$.

a) For $B \in M_n(F)$, let $\varphi_B : M_n(F) \to F$ by $\varphi_B(A) = \mathrm{Tr}(AB)$. Show φ_B is in the dual space of $M_n(F)$ and compute $\varphi_B(E_{k\ell})$ in terms of the entries of B, where $E_{k\ell}$ is the matrix in $M_n(F)$ with 1 in the (k, ℓ) position and 0 in all other positions.

→ First, I will show that φ_B is in the dual space of $M_n(F)$, i.e., $\varphi_B : M_n(F) \to F$ is F-linear.
Let $A, A' \in M_n(F)$. Then $\varphi_B(A + A') = \mathrm{Tr}((A + A')B) = \mathrm{Tr}(AB + A'B) = \mathrm{Tr}(AB) + \mathrm{Tr}(A'B) = \varphi_B(A) + \varphi_B(A')$ where the third equality follows from the fact that for two matrices $A, A' \in M_n(F)$, s.t. $A = (a_{ij})$, $A' = (a'_{ij})$, we have $A + A' = (b_{ij})$ where $b_{ij} = a_{ij} + a'_{ij}$, so $\mathrm{Tr}(A + A') = \sum a_{ii} + a'_{ii}$ $= \sum a_{ii} + \sum a'_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(A')$.
Also for $c \in F$, we have $\varphi_B(cA) = \mathrm{Tr}(cAB) = c\,\mathrm{Tr}(AB) = c\varphi_B(A)$, where the second equality follows from the fact that for $A = (a_{ij})$ and $c \in F$, $\mathrm{Tr}(cA) = \sum ca_{ii} = c\sum a_{ii} = c\,\mathrm{Tr}(A)$.
Thus, for $B \in M_n(F)$, we have φ_B is F-linear.
Now let $B \in M_n(F) = (b_{ij})$. Then $\varphi_B(E_{k\ell}) = \mathrm{Tr}(E_{k\ell}B)$. Multiplying $E_{k\ell}B$ we see that
$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ & & 1 & \\ 0 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & b_{k\ell} & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ b_{\ell 1} & b_{\ell 2} & \cdots & b_{\ell n} \\ 0 & & 0 & 0 \end{pmatrix}$$
where the resulting matrix has zeros everywhere except in the k^{th} row, where the entries are that of the ℓ^{th} column of B. Thus, $\varphi_B(E_{k\ell}) = \mathrm{Tr}(E_{k\ell}B) = b_{k\ell}$.

b) Show $M_n(F) \cong M_n(F)^\vee$ as vector spaces by $B \mapsto \varphi_B$. Concretely, this means each linear mapping $M_n(F) \to F$ has the form $A \mapsto \mathrm{Tr}(AB)$ for a unique $B \in M_n(F)$.

→ We claim that $\rho : M_n(F) \to M_n(F)^\vee$ by $\rho(B) = \varphi_B$ is F-linear. Let $B, B' \in M_n(F)$. Then $\rho(B + B') = \varphi_{B+B'}$ and $\rho(B) + \rho(B') = \varphi_B + \varphi_{B'}$. Now let $A \in M_n(F)$. Then
$\varphi_{B+B'}(A) = \mathrm{Tr}(A(B + B')) = \mathrm{Tr}(AB + AB') = \mathrm{Tr}(AB) + \mathrm{Tr}(AB') = \varphi_B(A) + \varphi_{B'}(A) = (\varphi_B + \varphi_{B'})(A)$.
Thus, $\varphi_{B+B'} = \varphi_B + \varphi_{B'}$. Also for $c \in F$, $\rho(cB) = \varphi_{cB}$ and $c\rho(B) = c\varphi_B$. Then for $A \in M_n(F)$, $\varphi_{cB}(A) = \mathrm{Tr}(AcB) = c\,\mathrm{Tr}(AB) = c\varphi_B(A)$, so that $\varphi_{cB} = c\varphi_B$, hence φ is F-linear. Computing the kernel we see $\ker(\rho) = \{B \in M_n(F) : \rho(B) = 0\} = \{B \in M_n(F) : \varphi_B = 0\}$. If $\varphi_B = 0$, then $\varphi_B(A) = 0\ \forall A \in M_n(F)$. From part (a) we know that if $A = E_{k\ell}$, then $\varphi_B(E_{k\ell}) = b_{\ell k}$. Thus, for any $k, \ell \in \{1, 2, \dots, n\}$ we have $b_{\ell k} = 0$. Hence, B must be the zero matrix which shows that ρ is 1-1.
Finally, we know $\dim(M_n(F)) = \dim(M_n(F)^\vee) = n^2 < \infty$ so that ρ is surjective. Hence, it is an isomorphism.

continued...

④ Let V be an n-dimensional vector space over a field F. From class, each hyperplane H in V (that is, a subspace with dimension $n-1$) corresponds to a unique one-dim. subspace of $V^\vee$: $H = \ker(\varphi)$ for a nonzero φ that's unique up to scaling. Inspired by this, let's say that hyperplanes $H_1, \dots, H_r$ in V are "linearly ind." when the corresponding one-dim. subspaces $V^\vee$ are lin. ind. in the sense that a nonzero vector chosen from each one-dim. subspace forms a lin. ind. set in $V^\vee$. Prove $\dim_F\left(\bigcap_{j=1}^{r} H_j\right) \geq n-r$ with equality if and only if $H_1, \dots, H_r$ are linearly ind. hyperplanes. (Hint: Write $H_j = \ker(\varphi_j)$ for nonzero $\varphi_j \in V^\vee$ and use the lin. map $\Phi: V \to F^r$ where $\Phi(v) = (\varphi_1(v), \dots, \varphi_r(v))$, whose kernel is $\bigcap_{j=1}^{r} H_j$.)

→ Let $\Phi: V \to F^n$ by $\Phi(v) = (\varphi_1(v), \varphi_2(v), \dots, \varphi_r(v))$. First we want to prove that $\dim_F\left(\bigcap_{j=1}^{r} H_j\right) \geq n-r$. From the first iso. thm., we have that $V/\ker\Phi \cong \mathrm{Im}(V) < F^r$, where the $\dim(\mathrm{Im}(V)) \leq r$. We have that $\dim(V) - \dim_F(\ker(\Phi)) \leq \dim(F^r)$, so $\dim(V) - \dim_F\left(\bigcap_{j=1}^{r} H_j\right) \leq \dim(F^r) \Rightarrow n-r \leq \dim_F\left(\bigcap_{j=1}^{r} H_j\right)$, and we are done.

· Now we want to show that $\{H_i\}$ is lin. ind. iff $\dim_F\left(\bigcap_{j=1}^{r} H_j\right) = n-r$. We know that $\{H_i\}$ is lin. ind. iff $\{\varphi_i\}$ is lin. ind.. We also know that equality for the other statement holds iff Φ is onto. So, we want to prove that $\{\varphi_i\}$ is lin. ind. iff Φ is onto.

If Φ is onto, then $\exists v_i$ s.t. $\Phi(v_i) = (0, \dots, 1, \dots, 0)$ where 1 is in the i^{th}. We see that $\varphi_i(v_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$. Now we want to prove that $\{\varphi_i\}$ is lin. ind.. Assume it is not. Then for $\alpha_1\varphi_1 + \dots + \alpha_n\varphi_n = 0$ $\exists$ at least one $\alpha_i \neq 0$, where $\alpha_i \in F$. Then we can subtract $\alpha_i\varphi_i$ from both sides and get $\alpha_1\varphi_1 + \dots + \alpha_n\varphi_n = -\alpha_i\varphi_i$. If we evaluate both sides at v_i, they will be equal, so we get that $\alpha_1\varphi_1(v_i) + \dots + \alpha_n\varphi_n(v_i) = -\alpha_i\varphi_i(v_i)$. This means that $0 = -\alpha_i$, so we have a contradiction. Therefore, $\{\varphi_i\}$ is lin. ind.

· If $\{\varphi_i\}$ is lin. ind., then we know that there are r of them. We know that $\dim(V) = \dim(V^\vee) = n$. Since the φ_i's are lin. ind., we can extend them to make a basis in $V^\vee$. We know that $V \cong V^\vee$. We can take $\{\varphi_1, \dots, \varphi_r, \varphi_{r+1}, \dots, \varphi_n\}$ as a basis of $V^\vee$ because $\{\varphi_i\}$ is lin. ind.. This corresponds to $\{v_1, \dots, v_r, \dots, v_n\}$ s.t. $\varphi_i(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$. This property holds when we restrict to r, so we get $\{\varphi_1, \dots, \varphi_r\}$ and $\{v_1, \dots, v_r\}$. The $\{v_1, \dots, v_n\}$ are linearly independent. $\dim \operatorname{span}\{v_1, \dots, v_r\} = r \Rightarrow \dim \mathrm{Im}(\Phi) = r$.

Therefore, Φ is onto.

Let F be a field, and let $Tr_0(2,F)$ be the three-dim. space of 2×2 matrices over $\mathbf{F}$ with trace 0: $Tr_0(2,F) = \{A \in M_2(F) : Tr(A) = 0\} = \left\{\begin{pmatrix} x & y \\ z & -x\end{pmatrix} : x,y,z \in F\right\}$. For $M \in GL_2(F)$, let γ_M be the linear map "conjugate by M" on $Tr_0(2,F)$: $\gamma_M(A) = MAM^{-1}$. (The map γ_M has values in $Tr_0(2,F)$ since $Tr(MAM^{-1}) = Tr(A)$, so when A has trace 0 so does MAM^{-1}).

a) Compute the matrix of γ_M with respect to the ordered basis $\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$ of $Tr_0(2,F)$. Your answer, of course, will depend on the entries of M. Write $M = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ and set $\Delta = ad-bc$. (Remark: Since $\gamma_M \in GL_2(Tr_0(2,F))$ and $\gamma_{M_1M_2} = \gamma_{M_1}\gamma_{M_2}$, picking a basis for $Tr_0(2,F)$ lets us view $M \mapsto \gamma_M$ as a very nonobvious group hom. $GL_2(F) \to GL_3(F)$.).

→ Let our ordered basis be $\mathcal{B} = \left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right\} = \{e_1, e_2, e_3\}$, as we are given. Let $M = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in GL_2(F)$, and $\Delta = ad-bc$. We want to compute $[\gamma_M]_{\mathcal{B}} = [[\gamma_M(e_1)]_{\mathcal{B}}\ [\gamma_M(e_2)]_{\mathcal{B}}\ [\gamma_M(e_3)]_{\mathcal{B}}]$.

First, we will compute $[\gamma_M(e_1)]_{\mathcal{B}}$:

$$\gamma_{M_1}(e_1) = \frac{1}{\Delta}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix}ad+bc & -2ab\\ 2cd & -(ad+bc)\end{pmatrix} = \frac{1}{\Delta}\left[(ad+bc)\begin{pmatrix}1&0\\0&-1\end{pmatrix} + (-2ab)\begin{pmatrix}0&1\\0&0\end{pmatrix} + (2cd)\begin{pmatrix}0&0\\1&0\end{pmatrix}\right],$$

so $[\gamma_M(e_1)]_{\mathcal{B}} = \frac{1}{\Delta}\begin{pmatrix}ad+bc\\-2ab\\2cd\end{pmatrix}$.

Now we will compute $[\gamma_M(e_2)]_{\mathcal{B}}$: $\gamma_M(e_2) = \frac{1}{\Delta}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix} \Rightarrow [\gamma_M(e_2)]_{\mathcal{B}} = \frac{1}{\Delta}\begin{pmatrix}-ac\\a^2\\-c^2\end{pmatrix}$.

Now we will compute $[\gamma_M(e_3)]_{\mathcal{B}}$: $\gamma_M(e_3) = \frac{1}{\Delta}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix} \Rightarrow [\gamma_M(e_3)]_{\mathcal{B}} = \frac{1}{\Delta}\begin{pmatrix}bd\\-b^2\\d^2\end{pmatrix}$.

So the matrix of γ_M with respect to the ordered basis $\mathcal{B}$ is:

$$[\gamma_M]_{\mathcal{B}} = \frac{1}{\Delta}\begin{pmatrix}ad+bc & -ac & bd\\ -2ab & a^2 & -b^2\\ 2cd & -c^2 & d^2\end{pmatrix}.$$

b) Let $W \subset Tr_0(2,F)$ be the subspace of matrices of the form $\begin{pmatrix}0&y\\0&0\end{pmatrix}$. When $M \in GL_2(F)$ has lower-left entry 0, so $M = \begin{pmatrix}a&b\\0&d\end{pmatrix}$, check $w \in W \Rightarrow \gamma_M(w) \in W$, so γ_M can be considered as a linear operator on the quotient space $Tr_0(2,F)/W$. Find the determinant of this linear operator on $Tr_0(2,F)/W$, in terms of the matrix entries a,b,d of M.

→ First we want to check that $w \in W$ implies that $\gamma_M(w) \in W$. Let $w = \begin{pmatrix}0&y\\0&0\end{pmatrix}$ and $M = \begin{pmatrix}a&b\\0&d\end{pmatrix}$, then $\gamma_M(w) = MwM^{-1} = \frac{1}{ad}\begin{pmatrix}a&b\\0&d\end{pmatrix}\begin{pmatrix}0&y\\0&0\end{pmatrix}\begin{pmatrix}d&-b\\0&a\end{pmatrix} = \begin{pmatrix}0&ay/d\\0&0\end{pmatrix} = \frac{a}{d}\begin{pmatrix}0&y\\0&0\end{pmatrix} \in W$, where d is nonzero because $M \in GL_2(F)$.

Now we want to find the determinant of this linear operator on $Tr_0(2,F)/W$. Since we are mod-ing out by W, we have a new basis $\mathcal{B}' = \left\{\overline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}, \overline{\begin{pmatrix}0&0\\1&0\end{pmatrix}}\right\} = \{\bar{e}_1, \bar{e}_3\}$. We can use part (a) to compute $[\overline{\gamma_M(e_1)}]_{\mathcal{B}'}$ and $[\overline{\gamma_M(e_2)}]_{\mathcal{B}'}$ as follows:

$\overline{\gamma_M(e_1)} = \frac{1}{\Delta}\begin{pmatrix}ad&0\\0&-ad\end{pmatrix} = \frac{1}{\Delta}(ad)\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, so $[\overline{\gamma_M(e_1)}]_{\mathcal{B}'} = \frac{1}{\Delta}\begin{pmatrix}ad\\0\end{pmatrix}$ and

$\overline{\gamma_M(e_3)} = \frac{1}{\Delta}\begin{pmatrix}bd&0\\d^2&-bd\end{pmatrix} = \frac{1}{\Delta}\left[(bd)\begin{pmatrix}1&0\\0&-1\end{pmatrix} + (d^2)\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]$, so $[\overline{\gamma_M(e_3)}]_{\mathcal{B}'} = \frac{1}{\Delta}\begin{pmatrix}bd\\d^2\end{pmatrix}$.

So our matrix is $[\gamma_M]_{\mathcal{B}'} = \frac{1}{ad}\begin{pmatrix}ad & bd\\0 & d^2\end{pmatrix} = \begin{pmatrix}1 & b/d\\0 & d/a\end{pmatrix}$, where $\det([\gamma_M]_{\mathcal{B}'}) = \frac{d}{a}$.

The determinant of this linear operator on $Tr_0(2,F)/W$ is $\frac{d}{a}$.

continued...

③ Let G be a finite group and V_G be the set of functions $f: G \to \mathbb{C}$. (This is all functions, not just group homomorphisms). Then V_G is a real vector space under pointwise addition of functions and pointwise multiplication by real scalars. (It is also a complex vector space, using complex scalars, but we won't use that here).

a) For $g \in G$, let $\delta_g : G \to \{0,1\}$ by $\delta_g(x) = \begin{cases} 0 & \text{if } x \neq g \\ 1 & \text{if } x = g \end{cases}$. Prove that $\{\delta_g, i\delta_g : g \in G\}$ is a basis of V_G as a real vector space, so $\dim_{\mathbb{R}}(V_G) = 2|G|$. (Hint: if $f \in V_G$ could be written as $\sum_{g \in G}(b_g\delta_g + c_g i\delta_g)$ for some b_g and c_g in $\mathbb{R}$, determine what each b_g and c_g would have to be in terms of values of f).

→ Let $f \in V_G$. Then we can write f as $f = \sum_{g \in G}(b_g\delta_g + c_g i\delta_g)$ for some $b_g, c_g \in \mathbb{R}$. To prove that $\mathcal{B} = \{\delta_g, i\delta_g : g \in G\}$ is a basis of V_G as a real vector space, we want to show that $\mathcal{B}$ is a spanning set and that $\mathcal{B}$ is linearly independent.

- If we take $f(x)$ s.t. $x \neq g$, we get $f(x) = \sum_{g \in G}(b_g\delta_g(x) + c_g i\delta_g(x)) = 0$. If we take $f(g)$, we get that $f(g) = \sum_{g \in G}(b_g\delta_g(g) + c_g i\delta_g(g)) = \sum_{g \in G}(b_g + ic_g)$, where $Re(f(g)) = b_g$ and $Im(f(g)) = c_g$.

* Start with general $f \in V_G$ and prove $f = \sum_{g \in G}(b_g\delta_g + c_g i\delta_g)$ when $b_g = Re(f(g))$, $c_g = Im(f(g))$ by evaluating both sides at $x \in G$ and get some values. Therefore, $\mathcal{B}$ is a spanning set.

- Assume that $\sum_{g \in G}(\delta_g + i\delta_g) = f$, where $f = 0$. For $h \in G$, we have $f(h) = \sum_{g \in G}(b_g\delta_g(h) + c_g i\delta_g(h))$ $= b_h\delta_h(h) + ic_h\delta_h(h) = b_h + ic_h$ must have coefficients be 0. Therefore, $\mathcal{B}$ is linearly independent.

b) For each $g \in G$, let $T_g : V_G \to V_G$ by $(T_g f)(x) = f(gx)$. Prove T_g is $\mathbb{R}$-linear <u>and</u> $T_g(\delta_h) = \delta_{g^{-1}h}$ $\forall g, h \in G$.

→ Let $f, f' \in V_G$. Then $(T_g(f+f'))(x) = (f+f')(gx) = f(gx) + f'(gx) = (T_g f)(x) + (T_g f')(x)$ $= (T_g f + T_g f')(x)$, and for $c \in \mathbb{R}$, $(T_g(cf))(x) = (cf)(gx) = c(f(gx)) = (c(T_g f))(x)$, which shows that T_g is $\mathbb{R}$-linear.

- Now let $g, h \in G$. Then $(T_g(\delta_h))(x) = \delta_h(gx) = \begin{cases} 1, & \text{if } gx = h \\ 0, & \text{otherwise} \end{cases}$ and $\delta_{g^{-1}h}(x) = \begin{cases} 1 & \text{if } x = g^{-1}h \\ 0 & \text{otherwise} \end{cases}$, but $gx = h \Rightarrow x = g^{-1}h$, so $T_g(\delta_h) = T_{g^{-1}h}$.

c) For all functions f_1 and f_2 in V_G, define $\langle f_1, f_2 \rangle = Re\left(\sum_{g \in G} f_1(x)\overline{f_2(x)}\right) \in \mathbb{R}$. Prove $\langle \cdot, \cdot \rangle$ is an inner product on V_G (check all conditions for an inner product are satisfied) and $\{\delta_g, i\delta_g : g \in G\}$ is an orthonormal basis of V_G. (Hint: $Re(z) = Re(\bar{z})$ for all $z \in \mathbb{C}$).

→ First we will show that $\langle \cdot, \cdot \rangle$ is bilinear. Let $f_1, f_2, f_3 \in V_G$ and $c \in \mathbb{R}$. Then $\langle f_1, f_2 + f_3 \rangle = Re\left(\sum_{g \in G} f_1(x)\overline{(f_2+f_3)(x)}\right) = Re\left(\sum_{g \in G} f_1(x)(\overline{f_2(x)} + \overline{f_3(x)})\right) = Re\left(\sum f_1(x)\overline{f_2(x)}\right) + Re\left(\sum f_1(x)\overline{f_3(x)}\right)$ $= \langle f_1, f_2 \rangle + \langle f_1, f_3 \rangle$. We also have $\langle f_1, cf_2 \rangle = Re\left(\sum_{g \in G} f_1(x)\overline{cf_2(x)}\right) = c\,Re\left(\sum_{g \in G} f_1(x)\overline{f_2(x)}\right) = c\langle f_1, f_2 \rangle$. Linearity in the first component follows almost exactly as above, thus $\langle \cdot, \cdot \rangle$ is bilinear.

- Now we will show that $\langle \cdot, \cdot \rangle$ is symmetric. Let $f_1, f_2 \in V_G$, then since $Re(z) = Re(\bar{z})$ $\forall z \in \mathbb{C}$ $\langle f_1, f_2 \rangle = Re\left(\sum_{g \in G} f_1(x)\overline{f_2(x)}\right) = Re\left(\overline{\sum f_1(x)\overline{f_2(x)}}\right) = Re\left(\sum \overline{f_1(x)} f_2(x)\right) = \langle f_2, f_1 \rangle$.

- Lastly, we will show that $\langle \cdot, \cdot \rangle$ is positive definite. Let $f_1 \in V_G$, then $\langle f_1, f_1 \rangle = Re\left(\sum_{g \in G} f_1(x)\overline{f_1(x)}\right)$ $= Re\left(\sum_{g \in G} |f_1(x)|^2\right) \geq 0$. This equation equals 0 when $f_1(x) = 0$ $\forall x \in G \Rightarrow f_1 = 0$.

Thus, we conclude that $\langle \cdot, \cdot \rangle$ is an inner product on V_G.

- Now we WTS that $\mathcal{B} = \{\delta_g, i\delta_g : g \in G\}$ is an orthonormal basis for V_G. Let $f, f' \in \mathcal{B}$ s.t. $f \neq f'$. Then we can write $f = \delta_{g_j}$ or $f = i\delta_{g_j}$ for some $g_j \in G$. Suppose $f = \delta_{g_j}$, then since $f' \neq f$, $f' = i\delta_{g_j}$ or $f' = \delta_{g_k}$ or $f' = i\delta_{g_k}$ for $j \neq k$. If $f' = i\delta_{g_j}$, then $\langle f, f' \rangle = Re\left(\sum_{g \in G} \delta_{g_j}(x)\overline{i\delta_{g_j}(x)}\right) = Re(\bar{i})$ $= 0$. If $f' = \delta_{g_k}$ for $j \neq k$, then $\langle f, f' \rangle = Re\left(\sum_{g \in G} \delta_{g_j}(x)\delta_{g_k}(x)\right) = 0$ since $\delta_{g_j}(x) = 1$ iff $\delta_{g_k}(x) = 0$ since $j \neq k$ so that $\delta_{g_j}(x)\delta_{g_k}(x) = 0$ for all $x \in G$. The same argument holds for $f' = i\delta_{g_k}$. Now if $f = i\delta_{g_j}$, then similar reasoning shows that $\langle f, f' \rangle = 0$ if $f \neq f'$. Notice that if $f = \delta_{g_j}$, then $\langle f, f \rangle = \sum_{x \in G}(\delta_{g_j}(x))^2$ (since δ_{g_j} is real-valued) $= 1$. If $f = i\delta_{g_j}$, then $\langle f, f \rangle = Re\left(\sum_{x \in G} i\delta_{g_j}(x)\overline{i\delta_{g_j}(x)}\right) = Re\left(\sum_{x \in G} i\delta_{g_j}(x)(-i)\delta_{g_j}(x)\right) = Re(i(-i)) = 1$.

Thus, $\mathcal{B}$ is an orthonormal basis of V_G.

ed...

For $g \in G$, prove $T_g^* = T_{g^{-1}}$ relative to the inner product in (c), so T_g is self-adjoint if $g = g^{-1}$.

→ We want to find the unique linear map $A: V_G \to V_G$ such that $\langle T_g f_1, f_2 \rangle = \langle f_1, A f_2 \rangle$. Note that for $f_1, f_2 \in V_G$, $\langle T_g f_1, f_2 \rangle = \mathrm{Re}\left(\sum_{x \in G} f_1(gx)\overline{f_2(x)}\right)$ and $\langle f_1, T_{g^{-1}} f_2 \rangle = \mathrm{Re}\left(\sum_{x \in G} f_1(x)\overline{f_2(g^{-1}x)}\right)$. But if $x \in G$, then $gx = h$ for some $h \in G$, hence $x = g^{-1}h$. Thus, $f_1(gx)\overline{f_2(x)} = f_1(h)\overline{f_2(g^{-1}h)}$ for some $h \in G$. Therefore, $\sum_{x \in G} f_1(gx)\overline{f_2(x)} = \sum_{x \in G} f_1(x)\overline{f_2(g^{-1}x)}$ since we are summing over all $x \in G$. More explicitly, for $g \in G$, the map $\rho_g: G \to G$ such that $\rho(x) = gx$ is a bijection since clearly if $y \in G$, then $\rho(g^{-1}y) = y$ and if $gx = gy$ then $x = y$ by left mult. by g^{-1}. Thus, the two sums above are equal. By the uniqueness of T_g^*, we must have that $T_g^* = T_{g^{-1}}$ and hence $T_g = T_g^*$ if $g = g^{-1}$.

e) If $g = g^{-1}$ then (d) and the spectral theorem imply that V_G has an orthonormal basis of eigenvectors for T_g. When $g \neq 1$, so g has order 2, prove all the functions $\left\{\frac{1}{\sqrt{2}}(\delta_h + \delta_{gh}), \frac{1}{\sqrt{2}} i(\delta_h + \delta_{gh}), \frac{1}{\sqrt{2}}(\delta_h - \delta_{gh}), \frac{1}{\sqrt{2}} i(\delta_h - \delta_{gh}) : h \in G/\{1, g\}\right\}$ are eigenvectors of T_g with eigenvalue ± 1. We pick h from coset representatives in $G/\{1,g\}$ since if h is replaced by gh then each of the four functions built from h is either unchanged or changes by a sign. This is an orthonormal basis, but calculating all the inner products could be tedious, so check a special case: the functions $(1/\sqrt{2})(\delta_h + \delta_{gh})$ for different $h \in G/\{1,g\}$ are orthonormal. (Hint: use (b) for eigenvectors and (c) for inner products).

→ To show this, we will take T_g of each function and show that it equals ± 1 of the function itself:

$T_g\left(\frac{1}{\sqrt{2}}(\delta_h + \delta_{gh})\right) = \frac{1}{\sqrt{2}}(T_g(\delta_h) + T_g(\delta_{gh})) = \frac{1}{\sqrt{2}}(\delta_{g^{-1}h} + \delta_{g^{-1}gh}) = \frac{1}{\sqrt{2}}(\delta_{gh} + \delta_h)$, and the rest are

$T_g\left(\frac{1}{\sqrt{2}}(i\delta_h + i\delta_{gh})\right) = \frac{1}{\sqrt{2}}(i\delta_{gh} + i\delta_h)$, $T_g\left(\frac{1}{\sqrt{2}}(\delta_h - \delta_{gh})\right) = \frac{-1}{\sqrt{2}}(\delta_{gh} + \delta_h)$, $T_g\left(\frac{1}{\sqrt{2}}(i\delta_h - i\delta_{gh})\right) = \frac{-1}{\sqrt{2}}(i\delta_{gh} + i\delta_h)$.

Note, we know $g^{-1} = g$ because g has order 2.

Therefore, we have proved that all the functions listed are eigenvectors of T_g with eigenvalue ± 1.

• Now we want to check a special case of showing this is an orthonormal basis. If $h_1 \neq h_2$, then we WTS that $(1/\sqrt{2})(\delta_{h_1} + \delta_{gh_1})$ is orthogonal to $(1/\sqrt{2})(\delta_{h_2} + \delta_{gh_2})$:

$$\left\langle \frac{1}{\sqrt{2}}(\delta_{h_1} + \delta_{gh_1}), \frac{1}{\sqrt{2}}(\delta_{h_2} + \delta_{gh_2}) \right\rangle = \frac{1}{2}\mathrm{Re}\left(\sum_{x \in G} ((\delta_{h_1} + \delta_{gh_1})(g))\overline{((\delta_{h_2} + \delta_{gh_2})(g))}\right) = \begin{cases} 1, & h_1 = g = h_2 \text{ or } h_1 = h_2 = 1 \\ 0, & \text{else.} \end{cases}$$

We can see that the inner product above only equals 1 when $h_1 = h_2$ (i.e., the inner product only equals 1 for each function with itself), but in the beginning we stated $h_1 \neq h_2$, therefore the inner product above always equals 0. Therefore, the two functions are orthonormal for different h.

* Use bilinearity to avoid tedious calculations:

$$\left\langle \frac{1}{\sqrt{2}}(\delta_h + \delta_{gh}), \frac{1}{\sqrt{2}}(\delta_h + \delta_{gh}) \right\rangle = \frac{1}{2}\left(\langle \delta_h, \delta_h \rangle + \langle \delta_h, \delta_{gh} \rangle + \langle \delta_{gh}, \delta_h \rangle + \langle \delta_{gh}, \delta_{gh} \rangle\right) = \frac{1}{2}(1 + 0 + 0 + 1) = \frac{2}{2} = 1. ✓$$