[...]ust 2005

① (a) Prove the Minimum Principle for harmonic functions, i.e., show that if u is harmonic in a region Ω and u attains a minimum at a point $z_0 \in \Omega$, then u is constant.

Pf. Let u be harmonic and suppose $\exists\, z_0 \in \Omega$ s.t. $u(z_0) = \min_{z\in\Omega} u(z)$.

Since u is harmonic, $\exists\, r_{z_0}$ such that $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt \quad \forall\, r < r_{z_0}$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0)\,dt$$

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0) - u(z_0 + re^{it})\,dt = 0, \text{ but } u(z_0 + re^{it}) \geq u(z_0) \ \forall t$$

By continuity, $u(z_0) = u(z_0 + re^{it}) \ \forall t \ \forall\, r < r_{z_0}$

Thus, $u(z) = u(z_0)$ in $B_{r_{z_0}}(z_0)$.

Consider $E = \{z \in \Omega : u(z) = u(z_0)\}$

$z_0 \in E$, so $E \neq \emptyset$

E is closed by continuity.

By the above argument, if $z \in E$, then $B_{r_z}(z) \subseteq E$, so E is open.

By connectedness, $E = \Omega$, so $u(z) = u(z_0) \ \forall z \in \Omega$. $\square$

Continued...

(b) Suppose f is analytic in the unit disk Δ and continuous in $\bar{\Delta}$, and $f(z)$ is real for $|z|=1$. Show that f is constant.

Pf: Assume f is nonconstant.

$f(z) = u(x,y) + iv(x,y)$.

By assumption $v(x,y) \equiv 0$ on $|z|=1$.

$v(x,y)$ is harmonic since it's the imaginary part of an analytic function. So $v(x,y)$ is continuous.

Since $\bar{\mathbb{D}}$ is compact, and by the minimum and maximum principle for harmonic functions, v attains a minimum and maximum on $|z|=1$.

Thus, $v \equiv 0$.

By the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$.

Thus, $u_x = 0, u_y = 0$.

Hence, $u(x,y) = c$ for some $c \in \mathbb{R}$.

Therefore, $f(z) = c$, so f is constant. $\square$

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(2) Let f and g be analytic and non-zero in a connected open set Ω. Suppose also that there exists a sequence of complex numbers $z_n \in \Omega$ so that $z_n \to p \in \Omega$ and for all positive integers n, $\frac{f'(z_n)}{f(z_n)} = \frac{g'(z_n)}{g(z_n)}$.

Show that there is a constant c so that $g = cf$.

Pf: Let $h_1(z) = \frac{f'(z)}{f(z)}$ and $h_2(z) = \frac{g'(z)}{g(z)}$.

h_1, h_2 are analytic on Ω by assumption.

$h_1(z_n) = h_2(z_n)$ $\forall n \in \mathbb{N}$, $z_n \to p \in \Omega$.

By the identity theorem, $h_1(z) = h_2(z)$ $\forall z \in \Omega$,

i.e., $\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)}$. ☆

We want to show $g = cf$, i.e., $\frac{g}{f} = c$.

In other words, we WTS $\left(\frac{g}{f}\right)' = 0$

$\Rightarrow \frac{fg' - gf'}{f^2} = 0$.

we know ☆ $f'g = fg'$, i.e., $f'g - fg' = 0$.

Therefore, $g = cf$.

□

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(4) Suppose f is an entire function, f is bounded for $Re(z) \geq 0$, and f' is bounded for $Re(z) \leq 0$. Prove that f is a constant.

Pf: Let γ = line from 0 to w, where $w \in$ left half plane.

Let $|f'(z)| \leq M$ on $Re(z) \leq 0$

$|f(z)| \leq N$ on $Re(z) \geq 0$

Notice $f(w) = f(0) + \int_\gamma f'(z)\,dz$

$$|f(w)| \leq |f(0)| + M|w|$$
$$\leq M(1+|w|)$$

Let $r > |w|$, by Cauchy's estimate,

$$|f''(w)| \leq \sup_{|z|=r} \frac{|f(z)|}{r^2} \leq \frac{\max\{N, M(1+|w|)\}}{r^2}$$
$$\leq \frac{\max\{N, M(1+r)\}}{r^2}$$

Let $r \to \infty$, then $|f''(w)| \to 0$.

So $f(z) = a + bz$, but $|f(z)| \to \infty$ as $|z| \to \infty$ if $b \neq 0$.

Thus, $b = 0$, so f is constant.

$\square$