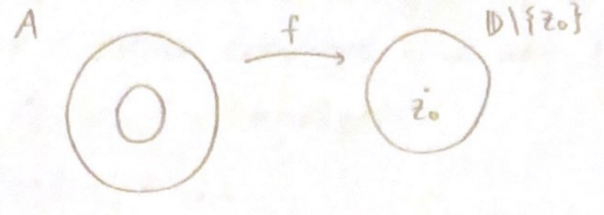
ust 2010

② Prove that there does not exist a 1-1 analytic function mapping an annulus onto a punctured disk.

Pf:

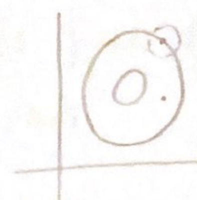


Assume such an f exists, i.e., $f: A \to \mathbb{D}\setminus\{z_0\}$ is 1-1 and analytic.

Then $f^{-1}: \mathbb{D}\setminus\{z_0\} \to A$ is also 1-1 and analytic.

Since f^{-1} is bounded near z_0, by Riemann's removable singularity thm, we have that z_0 is a removable singularity of f^{-1}.

Thus, f^{-1} extends to be analytic on all of $\mathbb{D}$.



So $f^{-1}(z_0) \in Int(A)$ since $f^{-1}(B_r(z_0)) \subseteq f(\mathbb{D})$ by the open mapping theorem.

Let $f^{-1}(z_0) = w \in Int(A)$.

Since f is 1-1 and onto, $\exists\, z_1$ s.t. $f^{-1}(z_1) = w$ s.t. $z_1 \neq z_0$.

Since $\mathbb{C}$ is Hausdorff, $\exists$ open nbhds U of z_0 and V of z_1 s.t. $U \cap V = \emptyset$.

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open.

So $f^{-1}(U) \cap f^{-1}(V)$ is open with $w \in f^{-1}(U) \cap f^{-1}(V)$.

Thus, $\exists$ nbhd B of w s.t. $B \subseteq f^{-1}(U) \cap f^{-1}(V)$.

Therefore, $\exists\, w' \in B$, $w' \neq w$ s.t. $w' \in f^{-1}(U)$, $w' \in f^{-1}(V)$

$\exists\, x_1 \in U$ s.t. $f^{-1}(x_1) = w'$ and $\exists\, x_2 \in V$ s.t. $f^{-1}(x_2) = w'$, $x_1 \neq x_2$

$f'(w)$ = not well-def. ↯

Therefore, there does not exist such an f.

□

ust 2010

④ Suppose the sequence $\{f_n\}$ of 1-1 analytic functions converges uniformly on compact subsets of a region Ω to a function f. Show that f is analytic, and is either constant or is also 1-1.

Pf: Claim: f is analytic.

- Let $z_0 \in \Omega$.

  Since Ω is open, $\exists\, r>0$ s.t. $B_r(z_0) \subseteq \Omega$.

  Let R be a rectangle in $B_r(z_0)$ and let γ be the curve parametrized by travelling around ∂R once in the positive direction.

  Then by uniform convergence $\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$.

  But $\int_\gamma f_n(z)\,dz = 0$ since f_n analytic in $B_r(z_0)$. So $\int_\gamma f(z)\,dz = 0$.

  Thus, by Morera's theorem, f is analytic in $B_r(z_0)$.

  Since z_0 is arbitrary, f is analytic in Ω.

Claim: f is constant or 1-1. (like Hurwitz: $f_n \to f$ unif. on cpt. subsets f_n never 0 in Ω, then $f \equiv 0$ or never 0 in Ω)

- Let $g_n(z) = f_n(z) - f_n(z_0)$ where $z_0 \in \Omega$.

  First, $g_n(z) \to f_n(z) - f_n(z_0)$ uniformly on compact subsets of Ω.

  g_n is analytic and 1-1 in Ω.

  $g_n(z) \neq 0$ in $\Omega \setminus \{z_0\}$ by injectivity.

  By Hurwitz, $f(z) - f(z_0) \equiv 0$ or never zero in $\Omega\setminus\{z_0\}$.

  If $f(z) - f(z_0) \equiv 0 \Rightarrow f(z) = f(z_0) \quad \forall\, z \in \Omega \setminus \{z_0\}$

  $f(z) \equiv f(z_0)$ in Ω (i.e. constant).

  If $f(z) - f(z_0) \neq 0 \;\forall\, z \in \Omega\setminus\{z_0\} \Rightarrow f(z) \neq f(z_0)$ in $\Omega \setminus \{z_0\}$.

  Thus, since z_0 is arbitrary, $f(z)$ is 1-1 in Ω.

$\square$

August 2010

(5) Let Ω be a bounded, simply connected domain in the plane. Suppose $g: \Omega \to \Omega$ is holomorphic and not the identity. Show that g can have at most one fixed point.

(a) First show it when Ω is the unit disc. Then

Pf: See Jan. 2019 #2.

(b) Show it when Ω is a bounded, simply connected region in the plane.

Pf: Since Ω is bounded, it is not all of $\mathbb{C}$, i.e., $\Omega \subsetneq \mathbb{C}$.
So by the Riemann mapping theorem, for any $z_0 \in \Omega$, $\exists!$ conformal map $f: \Omega \to \mathbb{D}$, onto, s.t. $f(z_1) = 0$, $f'(z_1) > 0$.

By contradiction, assume $g(z_1) = z_1$, $g(z_2) = z_2$ $(z_1 \neq z_2,\ z_1, z_2 \in \Omega)$

Let $f(z_2) = w_2$.

[We want a map $\mathbb{D} \to \mathbb{D}$
$f: \Omega \to \mathbb{D}$, $f^{-1}: \mathbb{D} \to \Omega$
$g: \Omega \to \Omega$

Then $f \circ g \circ f^{-1}(0) = f(g(z_1)) = f(z_1) = 0$
$f \circ g \circ f^{-1}(w_2) = f(g(z_2)) = f(z_2) = w_2$

Thus, $f \circ g \circ f^{-1}$ fixes two points.

So by part a, $f \circ g \circ f^{-1}(z) = z$.

Hence, $g(z) = z$. ↯

This is a contradiction since g is not the identity.

□

gust 2010

⑦ Prove that if f is a non-constant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Pf: Recall dense: for any $z_0 \in \mathbb{C}$, $\forall r > 0$ $\exists z \in \mathbb{C}$ s.t. $B_r(z_0) \cap f(z) \neq \emptyset$.

$(\exists z_n \in f(z)$ s.t. $z_n \to z_0)$

- Assume $f(\mathbb{C})$ is not dense in $\mathbb{C}$.

$\exists z_0 \in \mathbb{C}$ that is not a limit point of $f(\mathbb{C})$.

i.e., $\exists r > 0$ s.t. $B_r(z_0) \cap f(\mathbb{C}) = \emptyset$.

Then $|f(z) - z_0| \geq r > 0$ $\forall z \in \mathbb{C}$

So $f(z) - z_0 \neq 0$ $\forall z \in \mathbb{C}$.

Consider $g(z) = \dfrac{1}{f(z) - z_0}$. Notice $g(z)$ is analytic, in fact, g is entire since f is entire.

Also, $|g(z)| = \left| \dfrac{1}{f(z) - z_0} \right| \leq \dfrac{1}{r}$ $\forall z \in \mathbb{C}$

Since g is entire and bounded, by Liouville's theorem, we have that g is constant.

Therefore, $\dfrac{1}{f(z) - z_0} = c. \Rightarrow 1 = c(f(z) - z_0) \Rightarrow f(z) = \dfrac{1}{c} + z_0$

Hence, $f(z)$ is constant or $f(z) \equiv \infty$ (if $c = 0$) $\lightning$

$\hookrightarrow$ or f is not analytic

This is a contradiction to f being nonconstant.

Therefore, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

$\square$