ust 2016

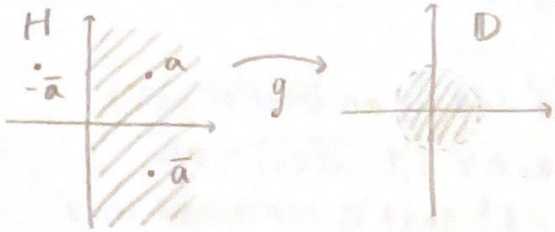
Prove the open mapping theorem: if $f: U \to \mathbb{C}$ is a non-constant analytic function defined on a connected, open set $U \subseteq \mathbb{C}$, then $f(V)$ is open for every open set $V \subseteq U$.

Pf: Suppose $V \subseteq U$ is open and let $w_0 \in f(V)$.
We WTS that $f(V)$ is open.
It suffices to show that every point of $f(V)$ is an interior point.
Since $w_0 \in f(V)$, there exists some $z_0 \in V$ s.t. $f(z_0) = w_0$.
We can find an open disk $\{z : |z - z_0| < \delta\}$ that is contained in V for some $\delta > 0$ since V is open.
By choosing a small δ, we can ensure the closed disk $\{z : |z - z_0| \leq \delta\}$ is contained in V.
By assumption f is non-constant analytic. Thus, $f(z) - w_0$ is not identically zero, and hence must have isolated roots.
In particular, we can choose δ so that $f(z) \neq w_0$ on the circle $C = \{z : |z - z_0| = \delta\}$.
Since $|f(z) - w_0|$ is a cts fn. that is never 0 on C and C is compact, there is some $\varepsilon > 0$ s.t. $|f(z) - w_0| \geq \varepsilon$ for all $z \in C$.
Fix a w s.t. $|w - w_0| < \varepsilon$ and consider $f(z) - w = \underline{f(z) - w_0} + \underline{w_0 - w}$.
By construction, we have on C that $|w - w_0| < \varepsilon \leq |f(z) - w_0|$.
So we can apply Rouché's theorem to conclude that $f(z) - w_0$ and $f(z) - w$ have the same number of roots inside C.
Since $z_0 \in C$ and $f(z_0) = w_0$, we see that $f(z) - w_0$ has at least one root inside C.
So there is some $z \in C$ s.t. $f(z) = w$.
Since $\{z : |z - z_0| < \delta\} \subset V$, it follows that
$w \in f(\{z : |z - z_0| < \delta\} \subset f(V) \Rightarrow w \in f(V)$.
Therefore, $f(V)$ is open for every open set $V \subseteq U$. $\square$

continued...

② Let $H := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ denote the right half plane. Prove that if $f : H$ is analytic and $f(H) \subseteq D(f(a), r)$ for some $a \in H$ and $r > 0$, then

$$\frac{|f(z) - f(a)|}{|z-a|} \leq \frac{r}{|z+\bar{a}|} \text{ for all } z \in H \setminus \{a\}, \text{ and } |f'(a)| \leq \frac{r}{2\mathrm{Re}(a)}.$$

Pf: 

First, since $a \in H$, we have that $a = x + iy$, where $x > 0$, so $\bar{a} = x - iy \in H$, and $-\bar{a} = -x + iy \in \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Let $g : H \to \mathbb{D}$ be given by $g(z) = \dfrac{z-a}{z+\bar{a}}$ if $|z-a| < |z+\bar{a}|$, then $z \mapsto \mathbb{D}$

so g is conformal. $g'(z) = \dfrac{z\bar{a}+a}{1-z}$

Let $h : D_r(f(a)) \to \mathbb{D}$ be given by $h(z) = \frac{1}{r}(z - f(a))$.

Then $h \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$, and

$$(h \circ f \circ g^{-1})(0) = h(f(g^{-1}(0))) = h(f(a)) = 0.$$

By Schwarz's lemma,

$$|(h \circ f \circ g^{-1})(z)| \leq |z|$$

$$|(h \circ f)(z)| \leq |g(z)| \Rightarrow |(h \circ f)(z)| \leq \left|\frac{z-a}{z+\bar{a}}\right|$$

$$\left|\frac{1}{r}(f(z) - f(a))\right| \leq \frac{|z-a|}{|z+\bar{a}|}$$

Thus, we have: $\dfrac{|f(z) - f(a)|}{|z-a|} \leq \dfrac{r}{|z+\bar{a}|}$

Letting $z \to a$, we have

$$|f'(a)| \leq \frac{r}{|a+\bar{a}|}, \text{ where } a = x+iy,\ \bar{a} = x - iy$$

$$(\text{so } |a+\bar{a}| = |2x| = 2\mathrm{Re}(a))$$

Therefore, $|f'(a)| \leq \dfrac{r}{2\mathrm{Re}(a)}$.

$\square$

...ued...

Let $A := \{z \in \mathbb{C} : r < |z| < R\}$ denote ~~the~~ an annulus, where $0 < r < R$. Prove that the function $f(z) := \frac{1}{z}$ cannot be uniformly approximated in A by complex polynomials.

Pf: Assume that $f(z) = \frac{1}{z}$ can be uniformly approx. in A by complex polys.

Let $f_n \to \frac{1}{z}$ uniformly.

Then $\int_C f_n \to \int_C \frac{1}{z}\,dz \neq 0$, where C is the contour around 0 inside A.

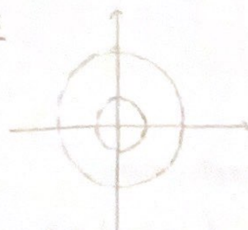
However, $\int_C f_n = 0$ for every n, since C is a closed curve and f_n is a polynomial. $\lightning$ Contradiction.

Therefore, $f(z) = \frac{1}{z}$ cannot be unif. approx. in A by complex poly.s.

$\square$

OR ☆

Pf:



uniformly approximated by polynomials means $\exists \{p_n\}$, p_n polynomials such that $p_n(z) \to f(z)$ uniformly in A.

Assume $\exists \{p_n\}$ polynomials s.t. $p_n(z) \to f(z)$ unif. in A

i.e., $\int p_n(z)\,dz \to \int f(z)\,dz$

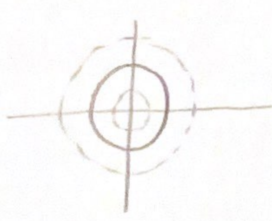
$\int p_n(z)\,dz \to \int \frac{1}{z}\,dz$

Recall: $\int_\gamma \frac{1}{z}\,dz = 2\pi i$, $\gamma = re^{it}dt$, $r > 0$

Let $\Gamma = se^{it}$, $0 \le t \le 2\pi$. We know $\int_\Gamma \frac{1}{z}\,dz = 2\pi i$

$r < s < R$

$\int_\Gamma p_n(z)\,dz = 0 \;\forall n$ $\lightning$ b/c $0 \neq 2\pi i$.



Therefore, $f(z) = \frac{1}{z}$ cannot be uniformly approximated in A by polynomials.

$\square$

continued...

④ Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ be a complex polynomial. Show that there must be at least one point with $|z| = 1$ and $|p(z)| \geq 1$.

Pf: Assume that all points with $|z|=1$ have $|p(z)|<1$.

Observe that in $p(z)$, $a_n = 1$, where a_n is the coefficient of z^n.

By Cauchy's integral formula, we have

$$a_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{p(z)}{z^{n+1}}\,dz.$$

Thus,
$$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=1} \frac{p(z)}{z^{n+1}}\,dz\right|$$
$$\leq \frac{1}{2\pi}\int_{|z|=1} \frac{|p(z)|}{|z|^{n+1}}\,|dz|$$
$$\leq \frac{1}{2\pi}\int_{|z|=1} |p(z)|\,|dz|$$
$$< \frac{1}{2\pi}\int_{|z|=1} |dz|$$
$$= \frac{2\pi}{2\pi} = 1.$$

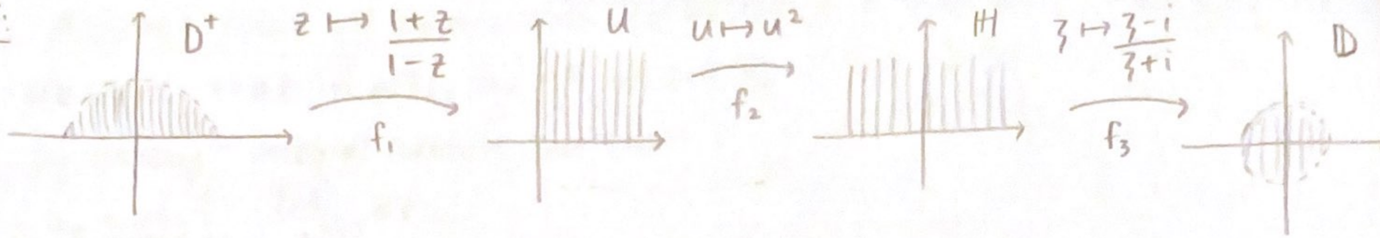
So we have that $a_n = 1$, and $|a_n| = 1 < 1$. ↯ Contradiction.

Therefore, there must be at least one point with $|z|=1$ and $|p(z)| \geq 1$.

Continued...

⑥ Find all 1-1 analytic maps from the upper half disk $D^+(0,1) := \{z \in \mathbb{C} : |z| < 1, \text{ and } \operatorname{Im}(z) > 0\}$ onto the unit disk $D(0,1)$.

Pf:



Let $f_1 : D^+ \to U$ be given by $f_1(z) = \frac{1+z}{1-z}$,

$f_2 : U \to \mathbb{H}$ be given by $f_2(u) = u^2$,

$f_3 : \mathbb{H} \to \mathbb{D}$ be given by $f_3(\zeta) = \frac{\zeta - i}{\zeta + i}$.

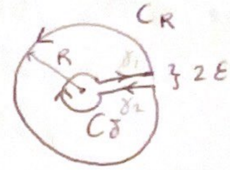
Let $f : U \to \mathbb{D}$ be given by $(f_3 \circ f_2 \circ f_1)(z)$.

We have that f is 1-1 analytic because the composition of 1-1 analytic maps is 1-1 analytic.

$\square$

ued.

Compute $\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$. Justify all manipulations. Hint: use the following contour:



Pf: Consider the following contour: $\Gamma = C_R \cup C_\delta \cup \gamma_1 \cup \gamma_2$.

Note that $f(z) = \frac{z^{1/3}}{z^2+1}$ is meromorphic in the region bounded by Γ by taking the branch of the logarithm such that $0 < \arg(z) < 2\pi$ $\left(z^{1/3} = e^{\frac{1}{3}\log z}\right)$

Thus, by the residue theorem,

$$\int_\Gamma \frac{z^{1/3}}{z^2+1}\,dz = 2\pi i\left[\operatorname{Res}_f(i) + \operatorname{Res}_f(-i)\right].$$

Let $z \in C_R$. Then

$$\left|\int_{C_R} \frac{z^{1/3}}{z^2+1}\,dz\right| \le \int_{C_R} \frac{|z|^{1/3}}{|z^2+1|}\,|dz|$$

$$\le \int_{C_R} \frac{|z|^{1/3}}{|z|^2-1}\,|dz|$$

$$= \int_{C_R} \frac{R^{1/3}}{R^2-1}\,|dz|$$

$$= \frac{R^{1/3}}{R^2-1}\cdot 2\pi R$$

Since this holds for all R large, letting $R\to\infty$, we see that $\left|\int_{C_R} f(z)\,dz\right| \to 0$.

Similarly, let $z \in C_\delta$, then

$$\left|\int_{C_\delta} \frac{z^{1/3}}{z^2+1}\,dz\right| \le \int_{C_\delta} \frac{|z|^{1/3}}{|z^2+1|}\,|dz|$$

$$\le \int_{C_\delta} \frac{|z|^{1/3}}{|z|^2-1}\,|dz|$$

$$= \frac{\delta^{1/3}}{\delta^2-1}\cdot 2\pi\delta \longrightarrow 0 \text{ as } \delta \to 0.$$

Therefore, $\lim\limits_{\substack{\delta\to 0\\ R\to\infty}} \int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

$$= 2\pi i\cdot\left[\operatorname{Res}_f(i) + \operatorname{Res}_f(-i)\right]$$

Continued...

Now on γ_1, $z = re^{ip}$ for some p very small.

Notice $z^{1/3} = e^{\frac{1}{3}\log(re^{ip})}$
$= e^{\frac{1}{3}(\log(r) + ip)}$
$= e^{\frac{1}{3}\log(r)} e^{ip/3}$.

So letting $\varepsilon \to 0$, we have $p \to 0$, so $z^{1/3} \to e^{\frac{1}{3}\log(r)} = r^{1/3}$.

Also on γ_2, $z = re^{i(2\pi - p)}$ for some p small, so by similar reasoning
$z^{1/3} \to e^{\frac{1}{3}(\log(r) + 2\pi i)} = r^{\frac{1}{3}} e^{2\pi i/3}$

Thus, on γ_2, $\int_{-\gamma_2} f(z)dz = \int_{\gamma_1} e^{2\pi i/3} f(z)dz$.

want γ_2 to run from 0 to ∞.

Hence, $-\int_{\gamma_2} f(z)dz = \int_{\gamma_1} e^{2\pi i/3} f(z)dz$

So $\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_1} e^{2\pi i/3} f(z)dz$
$= (1 - e^{2\pi i/3}) \int_{\gamma_1} f(z)dz$.

Then $\lim_{\substack{\delta \to 0 \\ R \to \infty}} (1 - e^{2\pi i/3}) \int_{\gamma_1} f(z)dz = (1 - e^{2\pi i/3}) \int_0^\infty f(z)dz$
$= 2\pi i \left[\mathrm{Res}_f(i) + \mathrm{Res}_f(-i)\right]$

Computing the residues, we get

$$\mathrm{Res}_f(i) = \lim_{z \to i} \frac{(z-i)z^{1/3}}{(z+i)(z-i)} = \lim_{z \to i} \frac{z^{1/3}}{z+i} = \frac{i^{1/3}}{2i} = \frac{e^{i\pi/6}}{2i}$$

$$\mathrm{Res}_f(-i) = \lim_{z \to -i} \frac{(z+i)z^{1/3}}{(z+i)(z-i)} = \lim_{z \to -i} \frac{z^{1/3}}{z-i} = \frac{(-i)^{1/3}}{-2i} = \frac{e^{i\pi/2}}{-2i}$$

Therefore, $\displaystyle\int_0^\infty f(z)dz = \frac{2\pi i \left[\frac{e^{i\pi/6} - e^{i\pi/2}}{2i}\right]}{(1 - e^{2\pi i/3})} = \frac{\pi}{\sqrt{3}}$.

$\square$