st 2017

① Suppose that $f$ is an entire function satisfying $f(n) = n$ for $n = 1, 2, \dots$ and $\lim_{|z|\to\infty} |f(z)| = \infty$. Show that $f(z) = z$.

Pf: Since $f$ is an entire function, we can write it as a convergent power series centered at $0$: $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Observe that $\lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to 0} \left|f\left(\frac{1}{z}\right)\right| = \infty$.

Notice that $f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ has a pole since $\lim_{|z|\to 0} \left|f\left(\frac{1}{z}\right)\right| = \infty$.

Since $f\left(\frac{1}{z}\right)$ has a pole, we have that $n=0$ after some $n = k > 0$,

so $f\left(\frac{1}{z}\right) = \sum_{n=0}^{k} \frac{a_n}{z^n} \Rightarrow f(z) = \sum_{n=0}^{k} a_n z^n$.

Since $f(m) = m$ for $m = 1, 2, \dots$, we have that $f(z) = z$.

□

Continued...

② Let $f$ be an analytic function on an open set containing the closure of $\mathbb{D}$, except for a simple pole at $z_0$ with $|z_0|=1$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series for $f$ in $\mathbb{D}$.

Show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

Pf: There exists a ball $B_\varepsilon(z_0)$ s.t. the function $(z-z_0)f(z)$ is holomorphic.
Observe that $B_\varepsilon(z_0) \cap \mathbb{D}$ is open since it is the finite intersection of open sets, and it is nonempty.
We have $f: \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic.
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $f$ is holomorphic and $(z-z_0)f(z)$ is holomorphic, we have that $\lim_{N\to\infty} (z-z_0)f(z) = \lim_{N\to\infty} (z-z_0)\sum_{n=0}^{N} a_n z^n$ exists, so each term $\sum (a_n - a_{n+1} z_0) z^{n+1}$ goes to 0.

$\lim_{n\to\infty} |a_n - a_{n+1} z_0||z^{n+1}| \to 0$

$\lim_{n\to\infty} |a_n - a_{n+1} z_0| \to 0$

$\lim_{n\to\infty} (a_n - a_{n+1} z_0) \to 0$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$ ✓

□

...ed...

Show that there does not exist an analytic $f: \mathbb{D} \to \mathbb{C}$ satisfying $\lim_{|z|\to 1} |f(z)| = \infty$.

Pf: Assume there exists such an analytic $f: \mathbb{D} \to \mathbb{C}$ satisfying $\lim_{|z|\to 1} |f(z)| = \infty$.

Then $g(z) = \frac{1}{f(z)}$ is a meromorphic function on $\mathbb{D}$.

We have $\lim_{|z|\to 1} |g(z)| = \lim_{|z|\to 1} \left|\frac{1}{f(z)}\right| = 0$.

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Pf: If $\lim_{|z|\to 1} |f(z)| = \infty$, then there exists $r$, $1 > r > 0$ such that $f \neq 0$ for $r < |z| < 1$.

Since $\overline{B_r(0)}$ is compact, $f$ can only have finitely many zeros in $\overline{B_r(0)}$ (otherwise they would have to have a limit in $\overline{B_r(0)}$ and the identity theorem would imply that $f \equiv 0$).

Let $z_1, \dots, z_k$ be the zeros of $f$ of orders $n_1, \dots, n_k$, respectively.

Let $g(z) = \frac{(z-z_1)^{n_1} \cdot \ldots \cdot (z-z_k)^{n_k}}{f(z)}$.

Then $g$ has removable singularities at $z_1, \dots, z_k$. Hence, $g$ extends to be analytic in $\mathbb{D}$.

Note that $\lim_{|z|\to 1} |g(z)| = 0$ by assumption.

So by the maximum principle, $g$ must be identically $0$.

But then $f(z) = \frac{(z-z_1)^{n_1} \cdot \ldots \cdot (z-z_k)^{n_k}}{g(z)}$ is not well-defined, and hence not analytic in $\mathbb{D}$. $\square$

continued...

(4) Let $f$ be an entire function satisfying that $\mathrm{Re} f - 4\,\mathrm{Im} f$ is bounded. Show that $f$ is constant.

Pf: We will use the fact that if $f$ and $g$ are entire functions, then $f \circ g$ is also an entire function.

Since $f$ is entire, so is $f + 4if$. Thus, $e^{f(z)+4if(z)}$ is entire.

Suppose $\mathrm{Re} f - 4\,\mathrm{Im} f \le M$ for some $M > 0$.

We have:

$$
\begin{aligned}
|e^{f(z)+4if(z)}| &= |e^{\mathrm{Re} f(z) + i\,\mathrm{Im} f(z) + 4i(\mathrm{Re} f(z) + i\,\mathrm{Im} f(z))}| \\
&= |e^{\mathrm{Re} f(z)} e^{i\,\mathrm{Im} f(z)} e^{4i\,\mathrm{Re} f(z)} e^{-4\,\mathrm{Im} f(z)}| \\
&= |e^{\mathrm{Re} f(z)}||e^{i\,\mathrm{Im} f(z)}||e^{4i\,\mathrm{Re} f(z)}||e^{-4\,\mathrm{Im} f(z)}| \\
&= |e^{\mathrm{Re} f(z)}||e^{-4\,\mathrm{Im} f(z)}| \\
&= e^{\mathrm{Re} f(z) - 4\,\mathrm{Im} f(z)} \\
&\le e^M.
\end{aligned}
$$

Since $e^{f(z)+4if(z)}$ is entire and bounded, by Liouville's theorem we have that it must be constant.

It remains to show that $f$ is constant.

We have: $(e^{f(z)+4if(z)})' = 0$ since $e^{f(z)+4if(z)}$ is constant

$$
\underbrace{(f'(z) + 4if'(z))}_{=0}\underbrace{e^{f(z)+4if(z)}}_{\neq 0} = 0
$$

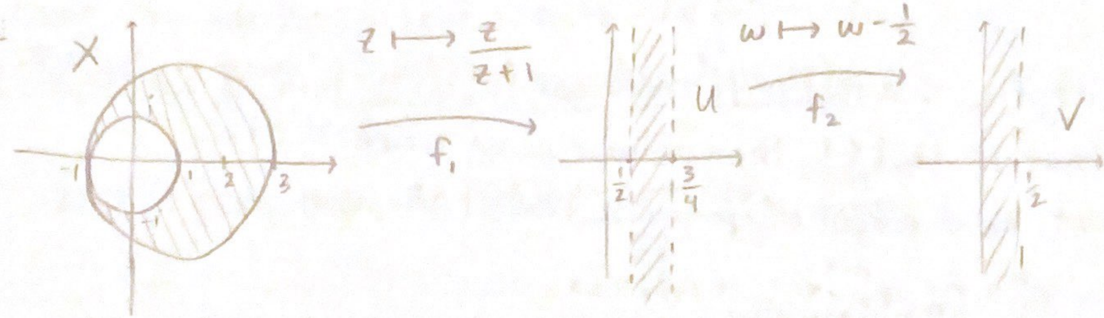
So $f'(z) + 4if'(z) = 0 \Rightarrow f'(z) = 0$.

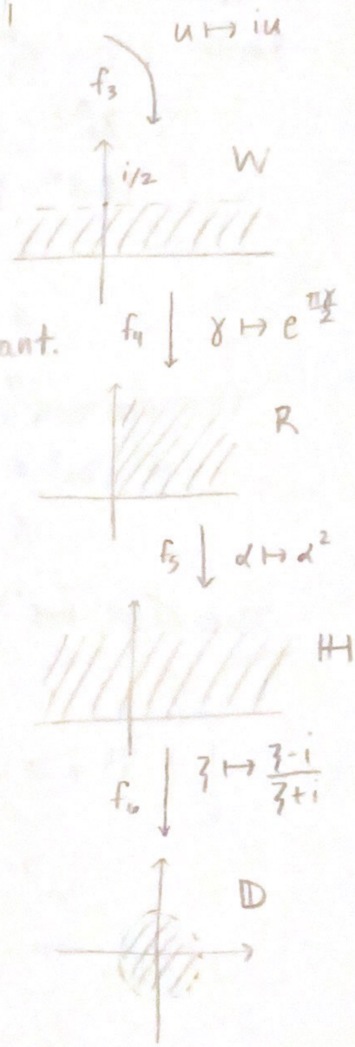
Therefore, $f$ is constant. $\square$

ded.

Find a conformal mapping from $\mathbb{D}$ onto open set bounded between $\{z : |z| = 1\}$ and $\{z : |z-1| = 2\}$.

Pf




Let $f_1 : X \to U$ by $f_1(z) = \frac{z}{z+1}$, so $-1 \mapsto \infty$, $1 \mapsto 1/2$, $3 \mapsto 3/4$

$f_2 : U \to V$ by $f_2(w) = w - \frac{1}{2}$, shifts the open strip.

$f_3 : V \to W$ by $f_3(u) = iu$, rotates the strip.

$f_4 : W \to R$ by $f_4(\gamma) = e^{\pi\gamma/2}$, maps to the first quadrant.

$f_5 : R \to \mathbb{H}$ by $f_5(\alpha) = \alpha^2$

$f_6 : \mathbb{H} \to \mathbb{D}$ by $f_6(\zeta) = \frac{\zeta - i}{\zeta + i}$

Let $f : X \to \mathbb{D}$ by $(f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(z)$

This is conformal b/c the composition of conformal maps is conformal.

$\square$

continued...

⑥ Let $p(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0$ be a complex polynomial, and let $R = \max(1, |c_0| + |c_1| + \ldots + |c_{n-1}|)$. Show that all the roots of $p$ are in $\{z : |z| \leq R\}$.

Pf: We WTS that if $z_0$ is a root of $p$, then $z_0 \in \{z : |z| \leq R\}$, that is, $|z_0| \leq R$.
First note that if $|z_0| \leq 1$, then we are done since $R \geq 1$.
Suppose that $z_0$ is a root of $p$ with $|z_0| > 1$.
Then,

$$0 = z_0^n + c_{n-1}z_0^{n-1} + \ldots + c_1 z_0 + c_0$$

$$\Rightarrow z_0^n = -(c_{n-1}z_0^{n-1} + \ldots + c_1 z_0 + c_0)$$

$$\begin{aligned}
|z_0|^n &= |c_{n-1}z_0^{n-1} + \ldots + c_1 z_0 + c_0| \\
&\leq |c_{n-1}||z_0|^{n-1} + \ldots + |c_1||z_0| + |c_0| \\
&\leq |c_{n-1}||z_0|^{n-1} + \ldots + |c_1||z_0|^{n-1} + |c_0||z_0|^{n-1} \\
&= |z_0|^{n-1}(|c_{n-1}| + \ldots + |c_1| + |c_0|)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |z_0| &\leq |c_{n-1}| + \ldots + |c_1| + |c_0| \\
&\leq R.
\end{aligned}$$

Therefore, we conclude that all roots of $p$ are in $\{z : |z| \leq R\}$.

$\square$

ed.

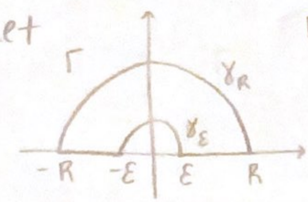
Use contour integration to show $\int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$.

(Hint: Show first that for real $z$, $\sin^2 z = \mathrm{Re}\, \frac{1-e^{2iz}}{2}$.)

Pf: First, let $f(z) = \frac{1-e^{2iz}}{2z^2} = \frac{1-\cos(2z)-i\sin(2z)}{2z^2}$.

If $z$ is real, then $\mathrm{Re}\, f(z) = \frac{1-\cos(2z)}{2z^2} = \frac{\sin^2 z}{z^2}$.

Let




$\Gamma$ be the contour we will use.

Then $\int_{-R}^{-\varepsilon} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{1-e^{2it}}{2t^2}\,dt$

$\begin{bmatrix}\text{let } z=\varepsilon e^{it} \\ dz=\varepsilon i e^{it}dt\end{bmatrix}$ $\int_{\gamma_\varepsilon} f(z)\,dz = \int_\pi^0 \frac{1-e^{2i\varepsilon e^{it}}}{2(\varepsilon e^{it})}\, i\,dt \xrightarrow[\varepsilon\to 0]{} \int_\pi^0 -i i\,dt = -\pi$ (L'Hospital's rule)

$\int_\varepsilon^R f(z)\,dz = \int_\varepsilon^R \frac{1-e^{2it}}{2t^2}\,dt$

$\int_{\gamma_R} f(z)\,dz = \int_0^\pi \frac{1-e^{2iRe^{it}}}{2(Re^{it})}\, i\,dt \xrightarrow[R\to\infty]{} 0$

By Cauchy's theorem, $\int_\Gamma f(z)\,dz \xrightarrow[\substack{\varepsilon\to 0\\ R\to\infty}]{} \int_{-\infty}^{\infty} \frac{1-e^{2it}}{2t^2}\,dt - \pi = 0$

By taking the real part, we get

$$\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{1-e^{2it}}{2t^2}\right) dt = \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\,du = 2\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \pi$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}. \qquad \square$$