ist 2018

1) How many roots (counted with multiplicity) does the function $f(z) = 5z^3 + e^z + 1$ have in the unit disk $\mathbb{D}$?

Pf: On $\partial\mathbb{D} \Rightarrow z=1$, we have $|5z^3| = 5$

$$\left.\begin{aligned} z &= e^{i\theta} = \cos\theta + i\sin\theta \\ e^z &= e^{\cos\theta + i\sin\theta} \\ |e^z| &= |e^{\cos\theta} e^{i\sin\theta}| \\ &= |e^{\cos\theta}| \approx 3 \end{aligned}\right\} \quad |e^z| \approx 3$$

$|1| = 1$

Let $p(z) = 5z^3$ and $g(z) = e^z + 1$.

On $\partial\mathbb{D}$, we have that

$$|g(z)| \leq 3 + 1 < 5 = |p(z)|.$$

Therefore, by Rouché's theorem, we have that p and $p+g$ have the same number of zeros in $\mathbb{D}$.

We know that $p(z) = 5z^3$ has a zero at $z=0$ with multiplicity 3.

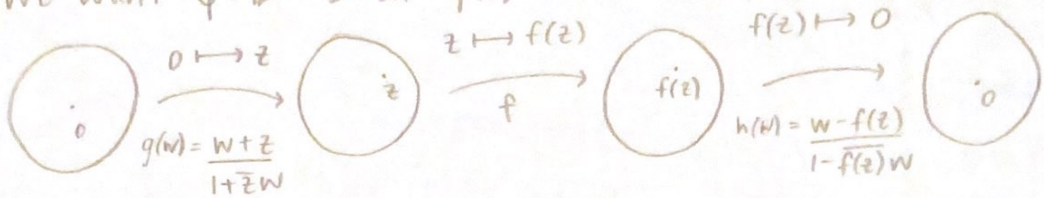
Thus, we conclude that $p(z) + g(z) = f(z) = 5z^3 + e^z + 1$ has three roots in $\mathbb{D}$. $\square$

continued...

② Let $f: \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \text{for all } z \text{ in } \mathbb{D}.$$

Pf: We want $\varphi: \mathbb{D} \to \mathbb{D}$ s.t. $\varphi(0)=0$.



Let $\varphi: \mathbb{D} \to \mathbb{D}$ s.t. $\varphi(z) = (h \circ f \circ g)(z) = h(f(g(z)))$

Then $\varphi(0)=0$, and $|\varphi(z)| \leq 1$.

By Schwarz lemma, we have that $|\varphi(z)| \leq |z|$ and $|\varphi'(0)| \leq 1$.

$$\varphi'(z) = h'(f(g(z)))f'(g(z))g'(z)$$

$$|\varphi'(0)| = |h'(f(g(0)))f'(g(0))g'(0)| \leq 1 \qquad g(0)=z$$

$$= |h'(f(z))f'(z)g'(0)|$$

$$= \left| \frac{1}{1-|f(z)|^2} \cdot f'(z) \cdot (1-|z|^2) \right| \leq 1$$

$$\Rightarrow \frac{|f'(z)|}{|1-|f(z)|^2|} \leq \frac{1}{|1-|z|^2|}$$

$$g(w) = \frac{w+z}{1+\bar{z}w}$$

$$g'(w) = \frac{(1+\bar{z}w)-(w+z)\bar{z}}{(1-\bar{z}w)^2}$$

$$g'(0) = \frac{(1+0)-(z)\bar{z}}{(1-0)^2} = \frac{1-|z|^2}{1} = 1-|z|^2$$

$$h(w) = \frac{w-f(z)}{1-\overline{f(z)}w}$$

$$h'(w) = \frac{(1-\overline{f(z)}w)-(w-f(z))(-\overline{f(z)})}{(1-\overline{f(z)}w)^2}$$

$$h'(f(z)) = \frac{(1-\overline{f(z)}f(z))-0}{(1-\overline{f(z)}f(z))^2} = \frac{1-|f(z)|^2}{(1-|f(z)|^2)^2} = \frac{1}{1-|f(z)|^2}$$

Therefore, we have shown that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \forall z \in \mathbb{D}.$$

□

...ued...

Let a be a real number. Suppose $f: \mathbb{C} \to \mathbb{C}$ is a holomorphic function that satisfies $\int_0^{2\pi} |f(re^{it})|\,dt \le r^a$ for all $r > 0$. Show that f is a polynomial.

Pf: Since f is entire, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Let $a \in \mathbb{R}$ and $r > 0$.

Then $a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz$. By taking the absolute value of a_n, we get

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz \right| \le \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |dz|$$

$$\le \frac{1}{2\pi r^{n+1}} \int_{|z|=r} |f(z)||dz| \qquad \left[\text{Let } z = re^{it} \Rightarrow dz = ire^{it}dt\right]$$

$$\le \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |f(re^{it})||ire^{it}dt|$$

$$\le \frac{r}{2\pi r^{n+1}} \int_0^{2\pi} |f(re^{it})||dt|$$

$$\le \frac{1}{2\pi r^n} \cdot r^a = \frac{r^{a-n}}{2\pi} = \frac{1}{2\pi r^{n-a}} \longrightarrow 0 \text{ as } r \to \infty \text{ if } n > a.$$

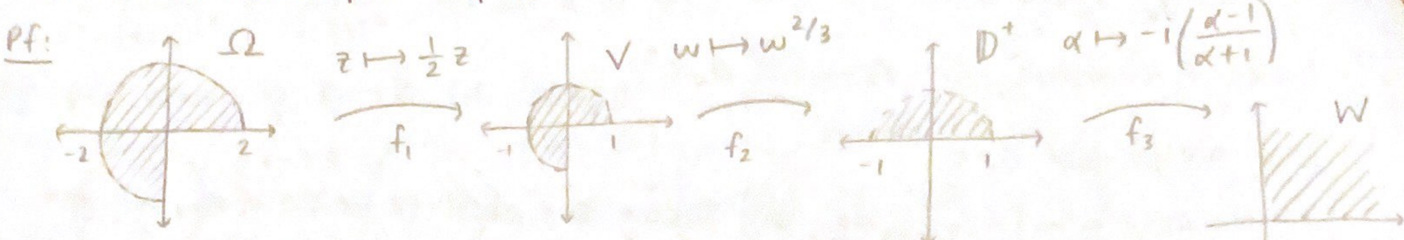
So $a_n = 0$ for $n > a$.

Therefore, f is a polynomial (of deg. $n < a$?). □

continued...

④ Let $\Omega = \{z = re^{i\theta} \in \mathbb{C} : 0 < r < 2 \text{ and } 0 < \theta < 3\pi/2\}$. Explicitly describe a one-to-one holomorphic map from Ω onto the unit disk $\mathbb{D}$.

Pf:



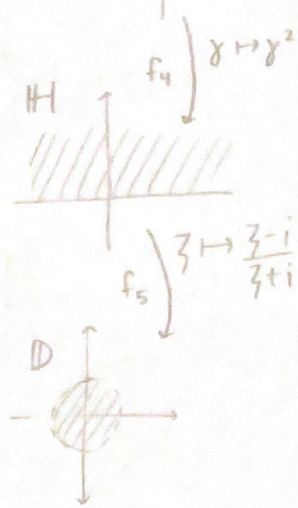
Let $f_1 : \Omega \to V$ by $f_1(z) = \frac{1}{2} z$

$f_2 : V \to \mathbb{D}^+$ by $f_2(w) = w^{2/3}$

$f_3 : \mathbb{D}^+ \to W$ by $f_3(\alpha) = -i\left(\frac{\alpha - 1}{\alpha + 1}\right)$

$f_4 : W \to \mathbb{H}$ by $f_4(\gamma) = \gamma^2$

$f_5 : \mathbb{H} \to \mathbb{D}$ by $f_5(\zeta) = \frac{\zeta - i}{\zeta + i}$.



Let $f : \Omega \to \mathbb{D}$ by $f(z) = (f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(z)$.

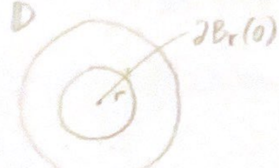
f is conformal and one-to-one because the composition of conformal maps is conformal and the comp. of 1-1 maps is 1-1.

$\square$

...ued...

Let f be a holomorphic function on $\mathbb{D}$. Suppose $|f(z)| \le |f(z^2)|$ for all $z \in \mathbb{D}$. Show that f is constant.

Pf: Choose $0<r<1$ and let $\partial B_r(0) = \{z \in \mathbb{C} : |z| \le r < 1\}$.



Since f is holomorphic on $\mathbb{D}$, by the maximum principle we have $M = \max_{z \in \overline{B_r(0)}} |f(z)|$ occurs on $\partial B_r(0)$.

Let $z_0 = re^{it_0}$, then $z_0^2 = r^2 e^{2it_0}$ $(r<1)$.

So $M = |f(re^{it_0})| \le |f((re^{it_0})^2)| = |f(r^2 e^{2it_0})|$

Since $r<1$, $r^2 < r$. So $(re^{it})^2 \in B_r(0)$.

By maximum modulus principle, f is constant in $B_r(0)$.

Letting $r \to 1$, we get that f is constant in $\mathbb{D}$.

(or we can use identity theorem). $\square$

OR:

Pf: Fix some r s.t. $0<r<1$.

Let $D_r = \{z \in \mathbb{C} : |z| < r\}$ (D_r is the open disk of radius r.)

Since f is holomorphic on $\mathbb{D}$ (and hence D_r), by the maximum principle $|f|$ attains a maximum on D_r.

Observe that for any $|z| = r$, $|z^2| \le |z|$ so $z^2 \in D_r$.

But we also have that $|f(z)| \le |f(z^2)|$.

This means that the maximum is attained within D_r.

Therefore, f is constant on D_r by the maximum principle.

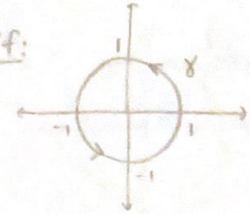
Letting $r \to 1$, we get that f is constant in $\mathbb{D}$.

$\square$

continued...

⑥ (1) Let $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, oriented in the counter-clockwise direction. Evaluate $\int_\gamma \frac{z^2+1}{z(z^2+4z+1)}\,dz$.

Pf:



First we will use the quadratic formula on z^2+4z+1:

$$z = \frac{-4 \pm \sqrt{16-4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}.$$

Observe that $-2-\sqrt{3} \notin \mathbb{D}$, but $-2+\sqrt{3} \in \mathbb{D}$.

Let $f(z) = \frac{z^2+1}{z(z^2+4z+1)}$. Then in $\mathbb{D}$, $f(z)$ has simple poles at $z=0$, $z=-2+\sqrt{3}$.

Computing the residues, we get:

$$\operatorname{Res}[f(z); z=0] = \lim_{z\to 0} (z)\cdot \frac{(z^2+1)}{z(z^2+4z+1)} = \lim_{z\to 0} \frac{z^2+1}{z^2+4z+1} = \frac{1}{1} = 1$$

$$\operatorname{Res}[f(z); z=-2+\sqrt{3}] = \lim_{z\to -2+\sqrt{3}} \frac{(z-(-2+\sqrt{3}))\,(z^2+1)}{z(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))}$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{(z^2+1)}{z(z+2+\sqrt{3})} = \frac{(-2+\sqrt{3})^2+1}{(-2+\sqrt{3})(-2+\sqrt{3}+2+\sqrt{3})}$$

$$= \frac{7-4\sqrt{3}+1}{(-2+\sqrt{3})(2\sqrt{3})} = \frac{8-4\sqrt{3}}{6-4\sqrt{3}}\cdot\frac{(6+4\sqrt{3})}{(6+4\sqrt{3})} = \frac{48+8\sqrt{3}-48}{36-48} = \frac{-8\sqrt{3}}{12} = \frac{-2\sqrt{3}}{3}.$$

By the residue theorem, $\int_\gamma \frac{z^2+1}{z(z^2+4z+1)}\,dz = 2\pi i\left(1-\frac{2\sqrt{3}}{3}\right) = 2\pi i - \frac{4\pi i\sqrt{3}}{3}$. □

(2) Evaluate $\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\,dx$. (Hint: You can use the contour integral in part (1).)

Pf: Let $z = e^{ix}$, then $dz = ie^{ix}dx \Rightarrow dx = \frac{dz}{iz}$. Also, $\cos(x) = \frac{e^{ix}+e^{-ix}}{2} = \frac{z+\frac{1}{z}}{2}$.

$$\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\,dx = \int_{|z|=1} \frac{\frac{1}{2}(z+\frac{1}{z})}{2+\frac{1}{2}(z+\frac{1}{z})}\cdot\frac{dz}{iz} = \int_{|z|=1} \frac{z+\frac{1}{z}}{4iz + iz(z+\frac{1}{z})}\,dz$$

$$= \int_{|z|=1} \frac{z+\frac{1}{z}}{4iz+iz^2+i}\,dz = \frac{1}{i}\int_{|z|=1} \frac{z^2+1}{4z^2+z^3+z}\,dz = \frac{1}{i}\underbrace{\int_{|z|=1} \frac{z^2+1}{z(z^2+4z+1)}\,dz}_{\text{this is the integral from part (1).}}$$

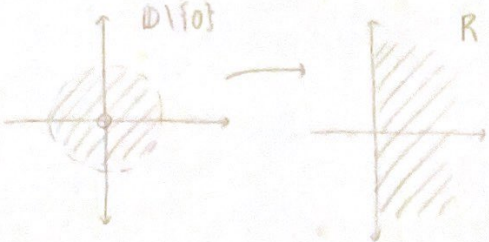
So we have $\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\,dx = \frac{1}{i}\left[2\pi i\left(1-\frac{2\sqrt{3}}{3}\right)\right] = 2\pi\left(1-\frac{2\sqrt{3}}{3}\right) = 2\pi - \frac{4\pi\sqrt{3}}{3}$.

□

...ued...

Suppose that $f(z)$ is holomorphic on the punctured unit disk $\mathbb{D}\setminus\{0\}$ and that the real part of f is positive. Prove that f has a removable singularity at 0.

Pf:



Let $g(z) = \dfrac{z-1}{z+1}$ map the right-half plane to the unit disk $\mathbb{D}$.

$f : \mathbb{D}\setminus\{0\} \longrightarrow R$ (Right half-plane)

$g : R \to \mathbb{D}$, $g^{-1} : \mathbb{D} \to R$ Thus, g^{-1} is defined. b/c g is a conformal map

Let $h(z) = (g \circ f)(z) : \mathbb{D}\setminus\{0\} \longrightarrow \mathbb{D}$.

Then h is analytic on $\mathbb{D}\setminus\{0\}$ and bounded on $\mathbb{D}\setminus\{0\}$, so $z=0$ is a removable singularity of $h(z)$. Hence Thus we can extend h to be analytic at $z=0$.

By the open mapping theorem, $h(0) \in \operatorname{Int}(\mathbb{D})$. So

let $h(0) = w \in \mathbb{D}$. Then $(g \circ f)(0) = g(f(0)) = w$. Since g is conformal,

Let $f(0) = g^{-1}(w)$. Then $z=0$ is a removable singularity of f. $\square$

Extra: Define $f(0) = g^{-1}(w)$

$\lim\limits_{z\to 0} h(z) = w \qquad \lim\limits_{z\to 0} (g \circ f)(z) = w$

$g\left(\lim\limits_{z\to 0} f(z)\right) = w$ by continuity

So $\lim\limits_{z\to 0} f(z) = g^{-1}(w)$

Hence f is extended to be analytic at $z=0$.