August 2019

① (a) Find all functions f which are holomorphic on $\mathbb{C}\setminus\{0\}$ and have the property that $z^2 f(z)$ is bounded on $\mathbb{C}\setminus\{0\}$.

Pf: Let $g(z) = z^2 f(z)$. $g: \mathbb{C}\setminus\{0\} \to \mathbb{C}$

The function g is analytic and bounded near $z=0$.

Therefore, by the Riemann Removable Singularity Theorem, we have that $z=0$ is a removable singularity.

Thus, g extends to be entire.

Since g is entire and bounded, by Liouville's Theorem, we have that g is constant: $g(z) = c \Rightarrow c = z^2 f(z) \Rightarrow f(z) = \frac{c}{z^2}$ for some constant $c \in \mathbb{C}$.

Therefore, $f(z) = \frac{c}{z^2}$.

$\square$

(b) Find all functions f which are holomorphic on $\mathbb{C}\setminus\{0\}$ and have the property that $z\sin(z)f(z)$ is bounded on $\mathbb{C}\setminus\{0\}$.

Pf: Let $g(z) = z\sin(z)f(z)$. $g: \mathbb{C}\setminus\{0\} \to \mathbb{C}$.

$$= z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) f(z)$$

$$= \left(z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^8}{7!} + \dots\right) f(z).$$

The function g is analytic and bounded near $z=0$.

Therefore, by the Riemann Removable Singularity Theorem, we have that $z=0$ is a removable singularity.

Thus, g extends to be entire.

Since g is entire and bounded, by Liouville's Theorem, we have that g is constant: $g(z) = c \Rightarrow c = z\sin(z)f(z) \Rightarrow f(z) = \frac{c}{z\sin(z)}$ for some constant $c \in \mathbb{C}$.

Note that $f(\pi) = \frac{c}{\pi\sin(\pi)} = \frac{c}{\pi \cdot 0} = \frac{c}{0}$.

In order to avoid this, our constant must equal 0, so

$f(z) = \frac{0}{z\sin(z)} = 0$.

Therefore, $f(z) = 0$.

$\square$

continued...

② Let $\gamma \subset \mathbb{C}$ be a positively-oriented simple closed curve not intersecting the set $\{-1, 1\}$. Compute all possible values of the integral $\int_\gamma \frac{2\,dz}{z^2-1}$, and give examples of curves γ which realize each value.

Pf: Let $f(z) = \frac{2}{z^2-1} = \frac{2}{(z+1)(z-1)}$.

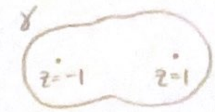
The function f has simple poles at $z = \pm 1$.

We can compute the residues at each pole:

$$\operatorname{Res}[f(z); z=1] = \lim_{z\to 1} (z-1)\cdot \frac{2}{(z+1)(z-1)} = \lim_{z\to 1} \frac{2}{z+1} = 1$$

$$\operatorname{Res}[f(z); z=-1] = \lim_{z\to -1} (z+1)\cdot \frac{2}{(z+1)(z-1)} = \lim_{z\to -1} \frac{2}{z-1} = -1.$$

• If γ is such that $z=\pm 1$ are in the area closed by γ as follows:

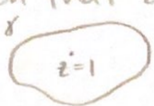


then by the residue theorem, we have:

$$\int_\gamma \frac{2\,dz}{z^2-1} = 2\pi i \sum_j \operatorname{Res}[f(z); z_j]$$
$$= 2\pi i\,[1+(-1)] = 2\pi i \cdot 0$$
$$= 0.$$

An example of such γ is $\gamma = \{z \in \mathbb{C} : |z| < 2\}$.

• If γ is such that $z=1$ is contained in the area enclosed by γ, and $z=-1$ is not, as follows:

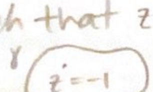


then by the residue theorem, we have:

$$\int_\gamma \frac{2\,dz}{z^2-1} = 2\pi i \cdot \operatorname{Res}[f(z); z=1]$$
$$= 2\pi i \cdot 1$$
$$= 2\pi i.$$

An example of such γ is $\gamma = \{z \in \mathbb{C} : |z-1| < 1\}$.

• If γ is such that $z=-1$ is contained in the area enclosed by γ, and $z=1$ is not, as follows:



then by the residue theorem, we have:

$$\int_\gamma \frac{2\,dz}{z^2-1} = 2\pi i \cdot \operatorname{Res}[f(z); z=-1]$$
$$= 2\pi i\,(-1)$$
$$= -2\pi i.$$

An example of such γ is $\gamma = \{z \in \mathbb{C} : |z+1| < 1\}$.

• If γ is such that it does not contain $z=1$ or $z=-1$, then by Cauchy's theorem we have that: $\int_\gamma \frac{2\,dz}{z^2-1} = 0$.

An example of such γ is $\gamma = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$.

□

...nued...

State and prove the Schwarz Lemma, including what ~~happens~~ occurs in the case of equality.

Pf: Schwarz Lemma: Let f be holomorphic in the unit disk $\mathbb{D}$ such that $|f(z)| \le 1$ and $f(0)=0$. Then we have that $|f(z)| \le |z|$ and $|f'(0)| \le 1$ for all $z \in \mathbb{D}$. If equality holds ($|f(z)| = |z|$), then we have that $f(z) = \lambda z$ where $|\lambda| = 1$, $\lambda \in \mathbb{C}$.

Proof: Let n be the order of the zero at $z=0$.

Then $g(z) = \dfrac{f(z)}{z^n}$ extends to be analytic in $\mathbb{D}$.

Let $r<1$. Since $\mathbb{D}$ is bounded and g is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, by the maximum principle we have that

$$\max_{z \in \overline{\mathbb{D}}} |g(z)| = \max_{z \in \partial \mathbb{D}} |g(z)| \Rightarrow \max_{|z| \le r} \left|\frac{f(z)}{z^n}\right| = \max_{|z|=r} \left|\frac{f(z)}{z^n}\right| \le \frac{|f(z)|}{r^n} \le \frac{1}{r^n} \to 1 \text{ as } r \to 1.$$

So we have $\left|\dfrac{f(z)}{z^n}\right| \le 1 \Rightarrow |f(z)| \le |z|^n \le |z|$ since $z \in \mathbb{D}$.

Therefore, $|f(z)| \le |z|$.

Suppose $|f(z_0)| = |z_0|$ for some $z_0 \ne 0$ in $\mathbb{D}$.

Then $1 = \dfrac{|f(z_0)|}{|z_0|} \le \dfrac{|f(z_0)|}{|z_0|^n} = |g(z_0)|$, so we have equality in the maximum principle, so $\forall z \in \mathbb{D}$, $g(z) = \lambda$ for some $|\lambda| = 1$, $\lambda \in \mathbb{C}$.

Then $f(z) = \lambda z^n$, but from $|f(z_0)| = |\lambda||z_0|^n \le |z_0|$

$|f(z_0)| = |z_0| \Rightarrow |z_0|^n = |z_0| \Rightarrow n = 1$.

Therefore, if equality holds, then we have $f(z) = \lambda z$ where $|\lambda| = 1$, $\lambda \in \mathbb{C}$.

$\square$

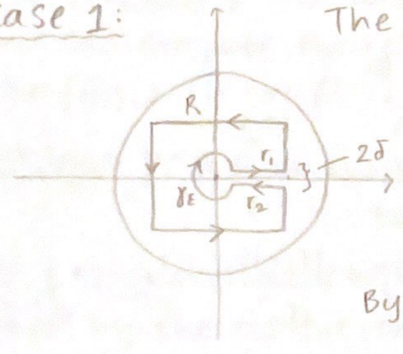
continued...

④ (a) Show that a continuous function $f: \mathbb{D} \to \mathbb{C}$ that is holomorphic on the slit disc $\mathbb{D} \setminus [0,1)$ is holomorphic on $\mathbb{D}$.

Pf: Let R be a rectangle in the unit disc $\mathbb{D}$ s.t. its sides are parallel to the real and imaginary axes.

- If R does not intersect $[0,1)$, then by Cauchy's theorem, $\int_{\partial R} f(z)dz = 0$.
- If R does intersect $[0,1)$, there are two possible cases:

Case 1:



The integral around the curve to the left is 0. Why?

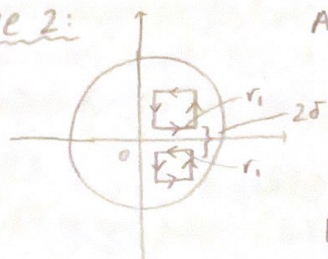
Since f is continuous, as $\delta \to 0$, then the integral over r_1, $\int_\varepsilon^1 f(z)dz$, and the integral over r_2, $\int_1^\varepsilon f(z)dz = -\int_\varepsilon^1 f(z)dz$, will cancel each other out when added.

By the ML inequality, $\left|\int_{\gamma_\varepsilon} f(z)dz\right| \leq \int_{\gamma_\varepsilon} |f(z)||dz| \to 0$ as $\varepsilon \to 0$.

So as $\varepsilon \to 0$ and $\delta \to 0$, the integral over the modified contour of R approaches the integral over R, which is also 0.

Therefore, by Morera's theorem, since we have that $\int_{\partial R} f(z)dz = 0$, we have that f must be holomorphic in $\mathbb{D}$.

Case 2:

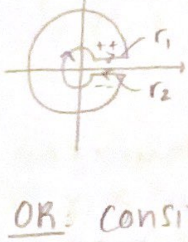


As $\delta \to 0$, by continuity we get that $\int_{r_1} + \int_{r_2} = \int_{r_1} - \int_{r_1} = 0$, and the modified curve R on the left becomes like . This new curve R' has $\int_{\partial R'} f(z)dz = 0$.

By Morera's theorem, we have that f must be holomorphic on $\mathbb{D}$. □

(b) Give an example of a holomorphic function $f: \mathbb{D} \setminus [0,1) \to \mathbb{C}$ that has no holomorphic extension to $\mathbb{D}$.

Pf: Let $f(z) = \log(z)$ with the branch cut on $[0, \infty)$.



Then $\int_{r_1} + \int_{r_2} \neq 0$ because f is not continuous on $[0,1)$.

OR: Consider $f(z) = \sqrt{-z}$, which is holomorphic on $\mathbb{D} \setminus [0,1)$. However, it cannot extend holomorphically to $\mathbb{D}$ (f is not continuous in $\mathbb{D}$). □

ued.

In this problem, $p_a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ is a cubic polynomial with coefficient vector $a = (a_0, a_1, a_2, a_3)$.

(a) State Rouché's Theorem.

Pf: Rouché's Theorem: Let f and g be two analytic functions in a bounded domain D. If $|g(z)| < |f(z)|$ on ∂D for all $z \in \mathbb{C}$, then f and $f+g$ have the same number of zeros in D, counting multiplicities. $\square$

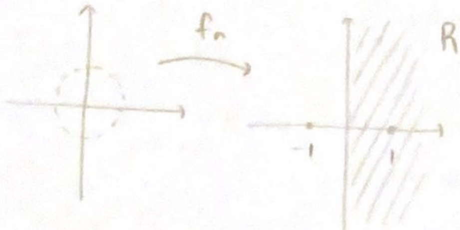
(b) The polynomial $p_{(1,1,1,1)}(z) = 1 + z + z^2 + z^3$ has a simple root at $z = -1$. Without using an explicit solution of the cubic, show that there is a neighborhood $U \subset \mathbb{C}^4$ of $(1,1,1,1)$ such that if $a \in U$, then $p_a(z)$ has a unique root $r(a)$ close to -1.

...nued...

Suppose f_n is a sequence of holomorphic functions on $\mathbb{D}$ such that $Re(f_n(z)) > 0$ for all $z \in \mathbb{D}$ and all n.

(a) If $f_n(0) = 1$ for all n, show that f_n has a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic f for which $Re(f(z)) > 0$ on $\mathbb{D}$

Pf:



R = right half plane

Let $g: R \to \mathbb{D}$ by $g(z) = \dfrac{z-1}{z+1}$.

$|g(z)| = \left|\dfrac{z-1}{z+1}\right|$

$|z-1| < |z+1| \Rightarrow \dfrac{|z-1|}{|z+1|} < 1 \;\forall z \in R.$

Let $g_n(z) = g \circ f_n : \mathbb{D} \to \mathbb{D}$.

Observe that $|g_n(z)| < 1$ for all $z \in \mathbb{D}$.

$\{g_n(z)\}$ is uniformly bounded on $\mathbb{D}$.

Therefore, it is a normal family.

So $\exists\, g_{n_k}(z)$ which converges uniformly on compact subsets of $\mathbb{D}$ to some analytic function h (on $\mathbb{D}$).

$g_{n_k} \to h \qquad |g_{n_k}(z_0) - h(z)| < \varepsilon$

$= g \circ f_{n_k} \to h \Rightarrow f_{n_k} \to g^{-1} \circ h$ ← b/c g is conformal

We have $h: \mathbb{D} \to \overline{\mathbb{D}}$

$f_n(0) = 1 \Rightarrow f_{n_k}(0) = 1 \Rightarrow g_{n_k}(0) = g \circ f_{n_k}(0) = g(1) = 0.$

$g_{n_k}(0) = 0 \;\forall n_k.$

Thus, $h(0) = 0$.

We have $h(z): \mathbb{D} \to \overline{\mathbb{D}}$, $h(0) = 0$.

So by Schwarz's lemma, $|h(z)| \le |z|$ for all $z \in \mathbb{D}$

$\Rightarrow |h(z)| < 1 \;\forall z \in \mathbb{D}$

Then $f_{n_k}(z) \to g^{-1} \circ h$ locally uniformly.

$Re(g^{-1} \circ h(z)) > 0.$ $\square$

continued...

(b) Is this true without the assumption that $f_n(0)=1$ for all n?

Pf: No.

Consider $f_n(z) = \frac{1}{n}$.

Then $f_n(z): \mathbb{D} \to H$, but $f_n(z) \to 0$ uniformly on $\mathbb{D}$. $\square$