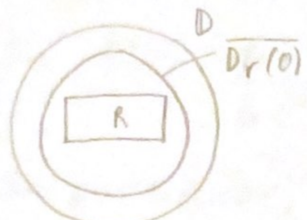
ast 2020

1) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a formal power series with complex coefficients. Prove that if the series converges for every $z \in \mathbb{D}$, then f is analytic in $\mathbb{D}$.

Pf: Since the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is at least 1, the series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{D_r(0)}$ for each $r<1$.

Let R be a rectangle in $\mathbb{D}$ and pick $r<1$ large enough that $R \subset \overline{D_r(0)}$.



The limit function, $f(z)$ is the uniform limit of continuous functions and is thus continuous (at least on $\overline{D_r(0)}$.)

Thus, we can write down $\int_{\partial R} f(z)\,dz$ without issue.

Furthermore, by uniform convergence, and the analyticity of z^n,

$$\int_{\partial R} f(z)\,dz = \int_{\partial R} \left(\sum_{n=0}^{\infty} a_n z^n\right) dz = \sum_{n=0}^{\infty} a_n \int_{\partial R} z^n\,dz = 0.$$

Thus, by Morera's theorem, f must be analytic on $\mathbb{D}$.

□

ued...

Let γ be a closed C^1 curve in $\mathbb{C}\setminus\mathbb{D}$ that winds around the origin twice in the counterclockwise direction. Compute $\int_\gamma \frac{8z^2-6z+1}{6z^2-5z+1}\,dz$. As always, justify your computation.

Pf: Note that the roots of $6z^2-5z+1=0$ are $z=\frac{1}{2},\frac{1}{3}$, which both lie in $\mathbb{D}$.

Note however that $8z^2-6z+1=0$ also has a root at $z=\frac{1}{2}$,

so $f(z)=\frac{8z^2-6z+1}{6z^2-5z+1}$ extends to be analytic at $z=\frac{1}{2}$.

In particular, $f(z)$ has a simple pole at $z=\frac{1}{3}$.

Computing the residue, we get

$$\operatorname{Res}[f(z), z=\tfrac{1}{3}] = \lim_{z\to 1/3} \frac{(z-\frac{1}{3})(8z^2-6z+1)}{3(z-\frac{1}{3})(2z-1)} = \lim_{z\to\frac{1}{3}} \frac{8z^2-6z+1}{3(2z-1)}$$

$$= \frac{8(\frac{1}{9})-2+1}{3(\frac{2}{3}-1)} = \frac{-\frac{1}{9}}{-1} = \frac{1}{9}$$

Therefore, by the residue theorem, we have that

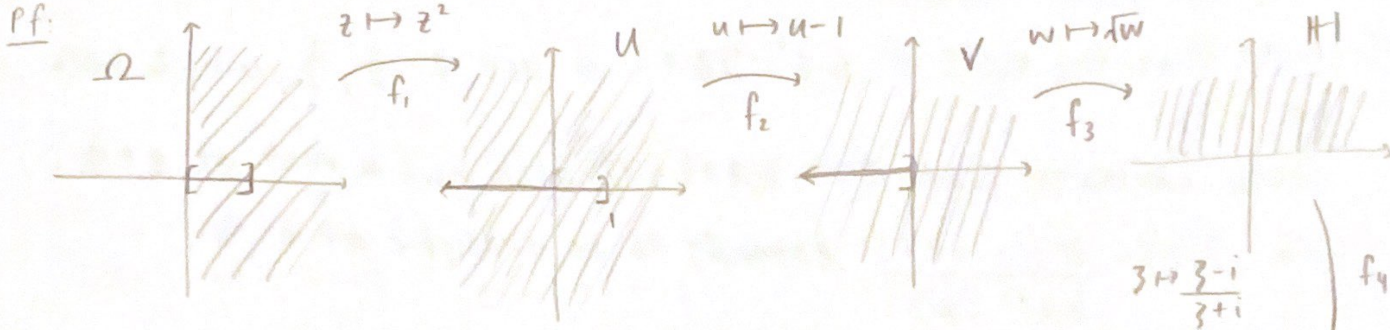
$$\int_\gamma \frac{8z^2-6z+1}{6z^2-5z+1}\,dz = 2\pi i\cdot 2\cdot\frac{1}{9} = \frac{4\pi i}{9}.$$

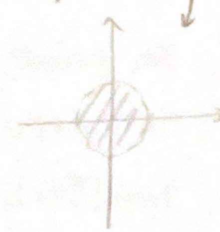
$\square$

continued...

④ Let $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0\} \setminus \{x + 0i : x \in (0, 1]\}$. Find a one-to-one analytic function $f : \mathbb{D} \to \Omega$ such that $f(\mathbb{D}) = \Omega$, $f(0) = 2$, and $f'(0) > 0$. You may describe f using a composition of maps.

Pf:





Let $f_1 : \Omega \to U$ by $f_1(z) = z^2$

$f_2 : U \to V$ by $f_2(u) = u - 1$

$f_3 : V \to \mathbb{H}$ by $f_3(w) = \sqrt{w}$

$f_4 : \mathbb{H} \to \mathbb{D}$ by $f_4(\zeta) = \dfrac{\zeta - i}{\zeta + i}$

Let $f : \Omega \to \mathbb{D}$ by $f(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z)$.

Observe that
$$\begin{aligned} f(0) &= f_4(f_3(f_2(f_1(0)))) \\ &= f_4(f_3(f_2(0))) \\ &= f_4(f_3(-1)) \\ &= f_4(i) \\ &= 0 \quad \checkmark \end{aligned}$$

nued...

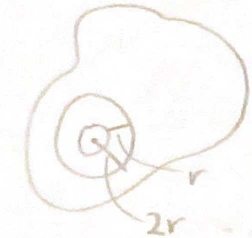
Let $\Omega \subseteq \mathbb{C}$ be a connected, open set. ~~and~~ Suppose that $f, f_1, f_2, \ldots : \Omega \to \mathbb{C}$ are analytic functions and $f_n \to f$ converges uniformly on compact sets. Prove that $f_n' \to f'$ uniformly on compact sets.

Pf: We WTS $|f_n'(z) - f'(z)| \to 0$ for all $z \in K$, $K \subseteq \Omega$ compact.

Let $\gamma = \partial B_{2r}(z)$, $z \in \Omega$, where $\overline{B_{2r}(z)} \subseteq \Omega$.

By Cauchy's formula, $f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\, d\zeta \quad \forall z \in B_r(z_0)$

$$|f_n'(z) - f'(z)| \le \frac{1}{2\pi}\int_\gamma \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2}|d\zeta| \le \frac{1}{2\pi r^2}\int_\gamma |f_n(\zeta) - f(\zeta)||d\zeta|$$



$|\zeta - z| > r$

$z \in B_r(z_0)$
$\zeta \in \partial B_{2r}(z_0)$

$$\le \frac{1}{2\pi r^2} \cdot \sup_{\zeta \in \gamma} |f_n(\zeta) - f(\zeta)| \cdot \underbrace{4\pi r}_{|d\zeta|}$$

By assumption, $\sup_{\zeta \in \gamma} |f_n(\zeta) - f(\zeta)| \to 0$ as $n \to \infty$.

$f_n' \to f$ uniformly in $B_r(z_0)$

Let K be a compact subset of Ω and cover K by finitely many disks $D_{r_1}, \ldots, D_{r_n}$ $(\overline{B_{2r_i}} \subseteq \Omega)$

for any $z \in K$, $|f_n'(z) - f'(z)| \le \frac{2}{r_0} \cdot \sup_{z \in K} |f_n(z) - f(z)|$

$r_0 = \min\{r_i : i = 1, \ldots, n\}$

$f_n' \to f'$ uniformly on K.

$\square$