gust 2021

① How many distinct $z \in \mathbb{D}$ satisfy $\frac{z^3+5z^2+1}{z+6} = 0$?

Pf: Note that $z=-6$ is not a zero of z^3+5z^2+1.

Let $\gamma = e^{it}$, $0 \le t \le 2\pi$.

On γ, we have that $|z^3| = 1$
$(|z|=1)$ $\quad |5z^2| = 5$
$\quad |1| = 1$.

Let $f(z) = 5z^2$ and $g(z) = z^3+1$.

On γ $(\partial\mathbb{D})$, we have that $|g(z)| \le 1+1 = 2 < 5 = |f(z)|$.

By Rouché's theorem, we have that f and $f+g$ have the same number of zeros in $\mathbb{D}$.

Since $f(z) = 5z^2$ has two zeros (zero at $z=0$ w/ mult. 2) in $\mathbb{D}$, we have that $f(z)+g(z) = z^3+5z^2+1$ also has two zeros in $\mathbb{D}$.

Now we will show that these two zeros are distinct.

Suppose that there is one zero z_0 of multiplicity 2 of z^3+5z^2+1 in $\mathbb{D}$.

Then $z^3+5z^2+1 = (z-z_0)^2 \cdot g(z)$, where $g(z)$ is nonzero at z_0 and analytic near z_0 (g is linear).

If a zero has multiplicity 2, then it must also be a zero of the derivative.

Taking the derivative, we get

$$3z^2+10z = 2(z-z_0)g(z) + (z-z_0)^2 g'(z).$$

Evaluating at z_0, we get $3z_0^2 + 10z_0 = 2(z_0-z_0)g(z_0) + (z-z_0)^2 g'(z_0)$
$= 0$

Note that $3z^2+10z = z(3z+10)$ has zeros at $z=0$ and $z = \frac{-10}{3}$, which are <u>not</u> zeros of z^3+5z^2+1.

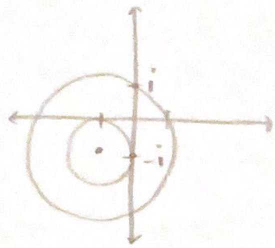
Therefore, $\frac{z^3+5z^2+1}{z+6} = 0$ has two distinct zeros in $\mathbb{D}$.

□

continued...

② Let $f(z) = \frac{1}{z^2+1}$, $\gamma_r = \{z \in \mathbb{C} : |z+i+1| = r\}$. Consider $g(r) = \int_{\gamma_r} f(z)\,dz$, where $r > 0$, and each γ_r is positively oriented. Find the domain of definition and the values of $g(r)$.

Pf: Note that $|z+i+1| = |z-(-i-1)|$.



The poles of $f(z)$ are $z = i$ and $z = -i$.
So $g(r)$ is undefined on the curves drawn on the left hand side.

If $z = i$ (big curve), then $|i+i+1| = |2i+1| = \sqrt{4+1} = \sqrt{5}$.
If $z = -i$ (small curve), then $|-i+i+1| = |1| = 1$.
Therefore, the domain of definition of $g(r)$ is $(0,1) \cup (1,\sqrt{5}) \cup (\sqrt{5},\infty)$.

- If $0 < r < 1$, then $f(z)$ is analytic in a nbhd of $\overline{B_r(-i-1)}$.
  Thus, by Cauchy's theorem, $g(r) = \int_{\gamma_r} f(z)\,dz = 0$.

- If $1 < r < \sqrt{5}$, then $f(z)$ has a pole at $z = -i$, so by the residue theorem,

$$g(r) = \int_{\gamma_r} f(z)\,dz = 2\pi i \cdot \operatorname{Res}_f(-i)$$
$$= 2\pi i \cdot \lim_{z \to -i} (z+i) \cdot \frac{1}{(z+i)(z-i)}$$
$$= 2\pi i \cdot \lim_{z \to -i} \frac{1}{z-i}$$
$$= 2\pi i \cdot \frac{1}{-2i}$$
$$= -\pi.$$

- If $r > \sqrt{5}$, then $f(z)$ has a pole at $z = i$ and $z = -i$, so by the residue theorem,

$$g(r) = \int_{\gamma_r} f(z)\,dz = 2\pi i \cdot (\operatorname{Res}_f(-i) + \operatorname{Res}_f(i))$$
$$= 2\pi i \left(\frac{1}{-2i} + \frac{1}{2i}\right)$$
$$= 2\pi i \cdot 0$$
$$= 0.$$

$\square$

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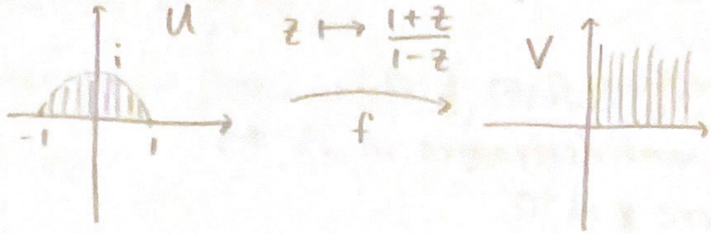
(5) Describe a holomorphic isomorphism between regions $U = \{z : |z| < 1, \operatorname{Im}(z) > 0\}$ and $V = \{z : \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\}$.

Pf:



Let $f : U \to V$ be given by $f(z) = \frac{1+z}{1-z}$.

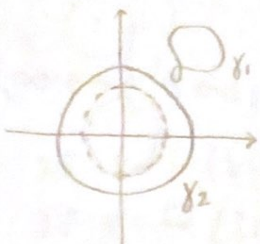
Then $f(z)$ is a conformal map from U onto V since $-1 \mapsto 0$, $1 \mapsto \infty$, and the angle at $z = -1$ is preserved. $\square$

continued.

(4) Let f be a rational function such that all its poles are contained in $\mathbb{D}$. Prove that f has an antiderivative outside of $\mathbb{D}$ if and only if the rational function $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$ has residue 0 at $z=0$.

Pf: · Suppose $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$ has residue 0 at $z=0$.

Recall that f has an analytic antiderivative in $\Omega \Leftrightarrow$ $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in Ω.



Case 1: Assume γ is a closed curve in $\mathbb{C}\setminus\mathbb{D}$ s.t. the region bounded by γ is contained in $\mathbb{C}\setminus\mathbb{D}$ (like γ_1).
Then $\int_\gamma f(z)\,dz = 0$ by Cauchy's theorem.

Case 2: Assume γ is a closed curve that winds around $\mathbb{D}$.
WLOG, we can assume $\gamma = Re^{it}$, $0 \le t \le 2\pi$, $R > 1$.

Note if z_i is a pole of f, then $|z_i| < 1$, so $\left|\frac{1}{z_i}\right| > 1$ for $z_i \neq 0$.
Also the poles of $g(z)$ are $z=0$ and the poles of $f\left(\frac{1}{z}\right)$
(because $f\left(\frac{1}{1/z_i}\right) = f(z_i)$, so $f\left(\frac{1}{z}\right)$ has poles at $\frac{1}{z_i}$).
Thus, $g(z)$ has poles outside of $\mathbb{D}$ except at $z=0$.
Letting $w = \frac{1}{z}$, $dw = \frac{-1}{z^2}dz$, we have

$$\int_\gamma f(w)\,dw = -\int_\Gamma f\left(\frac{1}{z}\right)\frac{1}{z^2}\,dz = -\int_\Gamma g(z)\,dz.$$

Notice if $w \in \gamma$, then $w = Re^{it}$, $R > 1$.
So $\frac{1}{z} = \frac{1}{R}e^{-it}$, $0 \le t \le 2\pi$, where $\frac{1}{R} < 1$.

Thus, $\Gamma = \frac{1}{R}e^{-it}$ is negatively oriented.

$-\int_\Gamma g(z)\,dz = \int_{-\Gamma} g(z)\,dz$, where $-\Gamma$ is positively oriented.

Therefore, by the Residue theorem, we have

$$\int_{-\Gamma} g(z)\,dz = 2\pi i \cdot \mathrm{Res}_g(0) = 2\pi i \cdot 0 = 0.$$

Hence, $\int_\gamma f(z)\,dz = 0$ for any closed curve $\gamma \subseteq \mathbb{C}\setminus\mathbb{D}$, so f has a primitive (analytic antiderivative) in $\mathbb{C}\setminus\mathbb{D}$.

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- Now suppose f has an analytic antiderivative outside of $\mathbb{D}$.

We WTS that $\mathrm{Res}_g(0) = 0$.

Let $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$.

We know from before $\int_\gamma f(z)\,dz = \int_{-\Gamma} g(z)\,dz$

$\gamma = re^{it}$ $\quad -\Gamma = \frac{1}{r} e^{it}$

$r > 1$ $\quad \frac{1}{r} < 1$

By the residue theorem,

$2\pi i \cdot \mathrm{Res}_g(0) = \int_c g(z)\,dz$, where c is a small circle around $z=0$ inside $\mathbb{D}$.

By assumption, $\int_\gamma f(z)\,dz = 0$.

Thus, $\int_c g(z)\,dz = 0 \Rightarrow 2\pi i \cdot \mathrm{Res}_g(0) = 0 \Rightarrow \mathrm{Res}_g(0) = 0$.

$\square$

continued...

⑤ Let f be a holomorphic function in a region. We say z_0 in the region is a local maximum for $|f|$ if $|f(z_0)| \geq |f(z)|$ for all z near z_0, and z_0 is a local minimum for $|f|$ when that inequality is reversed.

(a) Prove the strong maximum principle: f is constant if and only if $|f|$ has a local maximum.

Pf: · Suppose $|f|$ has a local maximum, i.e., $\exists\, z_0$ s.t. $|f(z_0)| \geq |f(z)|$ in $B_r(z_0)$ s.t. $\overline{B_r(z_0)}$ is contained in the region.

By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta, \text{ where } \gamma = z_0 + re^{it},\ 0 \leq t \leq 2\pi$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}} \cdot ire^{it}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt.$$

Thus, $|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|\,dt$

$$0 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})| - |f(z_0)|\,dt \qquad \left(\int_0^{2\pi} |f(z_0)|\,dt = 2\pi|f(z_0)|\right)$$

But $|f(z_0+re^{it})| - |f(z_0)| \leq 0$ since z_0 is a local maximum.

Therefore, $|f(z_0+re^{it})| = |f(z_0)|$.

Thus, $|f|$ is constant in $B_r(z_0)$.

Let $|f| = c$ in $B_r(z_0)$, where c is some constant.

Note that $f = u + iv$, where $u_x = v_y$ and $u_y = -v_x$. (Cauchy-Riemann)

Since $|f| = c$, $|f|^2 = u^2 + v^2 = c^2$

so differentiating, $2u \cdot u_x + 2v \cdot v_x = 0$ and $2u \cdot u_y + 2v \cdot v_y = 0$

$\Rightarrow 2uu_x - 2vu_y = 0$ and $2uu_y + 2vu_x = 0$

So $M = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note that $\det(M) = u^2 + v^2 = c^2$ by $|f|^2$.

If $c \neq 0$, then $\det(M) \neq 0$. Hence, $u_x = u_y = 0$, so u is constant.

By Cauchy-Riemann, v is also constant. Hence, f is constant.

If $c = 0$, then $|f| = 0$, so $f \equiv 0$. Hence, f is constant.

...nued...

Thus, f is constant in $B_r(z_0)$.
By the identity theorem, f is constant in all of the region.

- Suppose f is constant.
  Let $f(z) = c$ for some constant c.
  Then $|f(z)| = c$ $\forall z \in \Omega$ (our region).
  So $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$.
  Take any $z_0 \in \Omega$, we have that z_0 is a local maximum.
  Therefore, $|f|$ has a local maximum. $\square$

(b) Is it true that f is constant if and only if $|f|$ has a local minimum? Give a proof or a counterexample.

Pf: Let $f(z) = z$ in $\mathbb{D}$.

Then $|f|$ attains a minimum at $z=0$, but f is not constant. $\square$

Continued...

⑥ (a) Give definitions of a harmonic conjugate and of a normal family.

Pf: • Let u be harmonic (continuous, real-valued function s.t. $u_{xx}+u_{yy}=0$).
Then v is the harmonic conjugate of u if v is harmonic and $f=u+iv$ is analytic.

• Let $\mathcal{F}$ be a family of analytic functions on a domain Ω.
Then $\mathcal{F}$ is normal if for any sequence $\{f_n\} \subseteq \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that converges $\underbrace{\text{locally uniformly}}_{\text{converges uniformly on compact subsets of } \Omega}$.

□

* A family $\mathcal{F}$ is normal $\Leftrightarrow$ $\mathcal{F}$ is locally uniformly bounded:
for any compact subset $K \subseteq \Omega$, there exists C_K such that $|f| \le C_K$ for all $f \in \mathcal{F}$ on K.

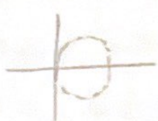
(holds for harmonic functions)

(b) Assume that u is a harmonic function in a simply connected region and v is its harmonic conjugate. Is it true that u is bounded if and only if v is bounded? Give a proof or a counterexample.

Pf: Consider $\operatorname{Log}(z) = \underbrace{\log|z|}_{u} + i\underbrace{\arg(z)}_{v}$. This is harmonic.

$\{z : -\pi < \arg(z) < \pi\}$

$\Omega = \{z : |z-1| < 1\}$



u is not bounded when we get close to 0.
but v is bounded. □

(c) Assume that $\mathcal{U} = \{u_n\}$ is a family of harmonic functions in a simply connected region and $\mathcal{V} = \{v_n\}$ is the family of their harmonic conjugates. Is it true that $\mathcal{U}$ is normal if and only if $\mathcal{V}$ is normal? Give proof or counterexample.

Pf: Let $\mathcal{U} = \{u_n = n\}$ and $\mathcal{V} = \{v_n = 0\}$.

We need $u_{xx}+u_{yy}=0$ and $u_x = v_y$, $u_y = -v_x$, and $(u_n)_x = 0$, $(u_n)_y = 0$.

$\mathcal{U}$ is not bounded, but $\mathcal{V}$ is bounded.

$\mathcal{U}$ is not a normal family, but $\mathcal{V}$ is normal.

□