uary 2014

1) Let G be a bounded connected open subset of $\mathbb{C}$, and let f be a nonconstant continuous function on $\overline{G}$ which is holomorphic on G. Assume that $|f(z)|=1$ for all $z \in \partial G$. Show that f has at least one zero in G.

Pf: Assume that f is nonzero in G, i.e., $f(z) \neq 0 \ \forall z \in G$.

Since f is holomorphic on G and continuous on $\overline{G}$, by the maximum principle, we have that $|f|$ attains a maximum on ∂G.

Since f is nonzero, by the minimum modulus principle, we have that $|f|$ attains a minimum on ∂G.

So we have $\max\limits_{z \in \overline{G}} |f(z)| = \min\limits_{z \in \overline{G}} |f(z)| = 1$ since $|f(z)|=1$ on ∂G.

So $|f(z)|=1$ for all $z \in G$.

Therefore, by the maximum principle, we get that f is constant in G. $\lightning$

This is a contradiction since we are given f is nonconstant in G.

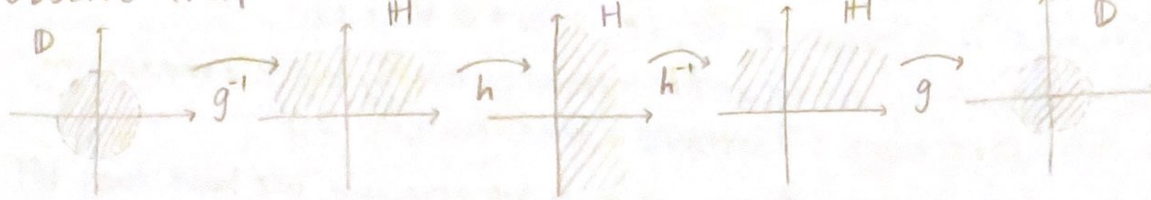
Thus, f has at least one zero in G. $\square$

continued...

② Let $f(z)$ be holomorphic in the right half-plane $H := \{z \in \mathbb{C} : Re(z) > 0\}$, with $|f(z)| < 1$ for all $z \in H$. If $f(1) = 0$, how large can $|f(2)|$ be?

Pf: We want $\varphi : \mathbb{D} \to \mathbb{D}$ s.t. $\varphi(0) = 0$ and $|\varphi(z)| \le 1$.

Observe that



Let $g(z) = \frac{z-i}{z+i}$, so $g^{-1}(z) = \frac{-i(1+z)}{z-1}$, and $h(z) = -iz$, so $h^{-1}(z) = iz$.

Notice that $f(z) = (g \circ h^{-1})(z)$ and $f(1) = g(h^{-1}(1)) = 0$, $|f(z)| < 1$. ✓

Let $\varphi : \mathbb{D} \to \mathbb{D}$ by $\varphi(z) = (g \circ h^{-1} \circ h \circ g^{-1})(z)$.

We have that $|\varphi(z)| < 1$ and $\varphi(0) = g(h^{-1}(h(g^{-1}(0)))) = g(h^{-1}(h(i))) = g(h^{-1}(1))$
$= f(1) = 0$ ✓

By Schwarz's lemma, we have that $|\varphi(z)| \le |z|$:

$$|\varphi(z)| = |(g \circ h^{-1} \circ h \circ g^{-1})(z)| \le |z|$$

$$|(f \circ h \circ g^{-1})(z)| \le |z| \Rightarrow |f(h(g^{-1}(z)))| \le |z|$$

$$\left|f\left(\frac{-(1+z)}{z-1}\right)\right| = \left|f\left(\frac{-1-z}{z-1}\right)\right| \le |z|$$

$$\Rightarrow |f(z)| \le \left|\frac{-z+1}{-z-1}\right|$$

So $|f(2)| \le \left|\frac{-2+1}{-2-1}\right| = \left|\frac{-1}{-3}\right| = \frac{1}{3}$.

Therefore, $|f(2)|$ can be as large as $\frac{1}{3}$.

Let $\psi(z) = \frac{-z+1}{-z-1}$. Then $|\psi(2)| = \left|\frac{-2+1}{-2-1}\right| = \left|\frac{-1}{-3}\right| = \frac{1}{3}$.

Thus, $\frac{1}{3}$ is a sharp bound. □

* ψ is a function that satisfies all the conditions of the hypothesis and $|\psi(2)| = \frac{1}{3}$. Thus, the upper bound is sharp (can be attained).

...nued...

Evaluate and justify your answer $\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$.

Hint: you can use the identity $a^6-1=(a^2-1)(a^4+a^2+1)$.

Pf: Using the hint, we can rewrite $\frac{x^2}{x^4+x^2+1}$ as $\frac{x^2(x^2-1)}{x^6-1}$.

Let $f(z)=\frac{z^2(z^2-1)}{z^6-1}$.

Notice that z^6-1 has roots $e^{i\pi/3}, e^{i2\pi/3}, e^{i\pi}, e^{i4\pi/3}, e^{i5\pi/3}, e^{2\pi i}$, and that z^2-1 has roots at $e^{i\pi}$ and $e^{2\pi i}$.

Therefore, $f(z)$ has simple poles at $z=e^{i\pi/3}, e^{i2\pi/3}, e^{i4\pi/3}, e^{i5\pi/3}$.

By the residue theorem, we have that $\int_\gamma f(z)dz = 2\pi i\cdot \sum_j \operatorname{Res}[f(z);z_j]$.

We will compute the residues:

$$\operatorname{Res}[f(z);z=e^{i\pi/3}] = \lim_{z\to e^{i\pi/3}} \frac{z^2}{(z-e^{2\pi i/3})(z-e^{4\pi i/3})(z-e^{5\pi i/3})} = \frac{e^{2\pi i/3}}{(e^{-i\pi/3})(e^{-\pi i})(e^{-4\pi i/3})}$$
$$= e^{10\pi i/3} = e^{4\pi i/3} = \frac{-1}{2}-\frac{i\sqrt{3}}{2}$$

$$\operatorname{Res}[f(z);z=e^{2\pi i/3}] = \lim_{z\to e^{2\pi i/3}} \frac{z^2}{(z-e^{i\pi/3})(z-e^{4\pi i/3})(z-e^{5\pi i/3})} = \frac{e^{4\pi i/3}}{(e^{\pi i/3})(e^{-2\pi i/3})(e^{-\pi i})}$$
$$= e^{8\pi i/3} = e^{2\pi i/3} = \frac{-1}{2}+\frac{i\sqrt{3}}{2}$$

$$\operatorname{Res}[f(z);z=e^{4\pi i/3}] = \lim_{z\to e^{4\pi i/3}} \frac{z^2}{(z-e^{i\pi/3})(z-e^{2\pi i/3})(z-e^{5\pi i/3})} = \frac{e^{8\pi i/3}}{(e^{\pi i})(e^{2\pi i/3})(e^{-\pi i/3})}$$
$$= e^{4\pi i/3} = \frac{-1}{2}-\frac{i\sqrt{3}}{2}$$

$$\operatorname{Res}[f(z);z=e^{5\pi i/3}] = \lim_{z\to e^{5\pi i/3}} \frac{z^2}{(z-e^{i\pi/3})(z-e^{2\pi i/3})(z-e^{4\pi i/3})} = \frac{e^{10\pi i/3}}{(e^{4\pi i/3})(e^{\pi i})(e^{\pi i/3})}$$
$$= e^{2\pi i/3} = \frac{-1}{2}+\frac{i\sqrt{3}}{2}.$$

Therefore, we have
$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i\left[\frac{-1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right]$$
$$= 2\pi i(-2)$$
$$= -4\pi i$$

Since f is an even function, we have that $\int_0^\infty f(z)dz = \frac{1}{2}\int_{-\infty}^{\infty} f(z)dz$.

Thus, we conclude that $\int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}(-4\pi i) = -2\pi i$.

$\square$

Continued...

(4) Suppose f is entire, and $\int_0^{2\pi} |f(re^{it})|\,dt \le r^{13/4}$ for all $r>0$. Prove that $f \equiv 0$.

Pf: Since f is entire, we can write f as a convergent power series centered at $z=0$: $f(z) = \sum_{n=0}^{\infty} a_n z_n$.

Fix $r>0$.

By Cauchy's integral formula, $a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz$.

Thus, $$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz\right| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}}|dz| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{r^{n+1}}|dz|$$

$$= \frac{1}{2\pi r^{n+1}}\int_{|z|=r} |f(z)||dz| = \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} |f(re^{it})||rie^{it}dt|$$

[Let $z=re^{it} \Rightarrow dz = rie^{it}dt$]

$$\le \frac{r}{2\pi r^{n+1}}\int_0^{2\pi} |f(re^{it})||dt|$$

$$\le \frac{1}{2\pi r}\cdot r^{13/4}$$

$$= \frac{r^{13/4-n}}{2\pi}.$$

So we have: $\frac{r^{\frac{13}{4}-n}}{2\pi} \to 0$ as $r\to\infty$ for $n>13/4$

$\frac{r^{\frac{13}{4}-n}}{2\pi} \to 0$ as $r\to 0$ for $n<13/4$

Therefore, $a_n = 0$ for all $n\in\mathbb{Z}$.

Thus, we conclude that $f\equiv 0$.

□

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Prove that there is no function f that is holomorphic in the punctured disk $D \setminus 0$, and f' has a simple pole at 0.

Pf. Assume such an f exists.

Since f' has a simple pole at 0, we have $f'(z) = \frac{c_{-1}}{z} + \sum_{n=0}^{\infty} c_n z^n$.

Since f is holomorphic in $\mathbb{D} \setminus \{0\}$ it has Laurent series
$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ (centered at $z=0$).

Therefore, $f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1}$.

But $\sum_{n=-\infty}^{\infty} n a_n z^{n-1} = \dots + \frac{2a_{-2}}{z^3} - \frac{a_{-1}}{z^2} + \frac{0 \cdot a_0}{z} + a_1 + 2a_2 z + \dots$,

which means that $c_{-1} = 0 \cdot a_0 = 0$. $\lightning$ Contradicts f' having a simple pole at $z=0$.

Therefore, such a function does not exist. $\square$

continued...

⑥ Let f be a complex-valued function in the unit disk D such that both functions f^2 and f^3 are holomorphic in D. Prove that f is holomorphic as well.

Pf: Suppose that f^2 and f^3 are holomorphic in $\mathbb{D}$.

Then $g = \frac{f^3}{f^2}$ is well-defined and holomorphic except at the zeros of f^2.

Note that if z_0 is a zero of f^2, it is a zero of f^3, as well as f.

Then $f^2(z) = (z-z_0)^m g(z)$ for some $m, n \in \mathbb{Z}$, and g, h holomorphic

$f^3(z) = (z-z_0)^n h(z)$ functions which are nonvanishing around z_0.

Observe that z_0 is also a zero of the holomorphic function $f^6 = (f^2)^3 = (f^3)^2$ of order $3m = 2n$.

There exists $k \in \mathbb{Z}^+$ s.t. $m = 2k$, $n = 3k$.

Hence, $\tilde{f}(z) = (z-z_0)^k \frac{h(z)}{g(z)}$ is well-defined and holomorphic around z_0.

When $z \neq z_0$, $\tilde{f}(z) = f(z)$. And $\tilde{f}(z_0) = 0 = f(z_0)$.

Therefore, $\tilde{f} = f$ is holomorphic at each zero z_0.

For anywhere else, $f = \frac{f^3}{f^2}$ is also holomorphic as $\frac{1}{f^2}$ is holomorphic.

□

f has a removable singularity at z_0 since it's bounded near z_0 (since it equals $\tilde{f}$ near z_0 and $\tilde{f}$ is holomorphic at z_0).

Then by continuity we must have $f = \tilde{f}$ at z_0.