...uary 2015

① Let $f(z) = \frac{1}{1+z^2} - \frac{z}{2-z}$. Write all of the Laurent series representations in z for the function f. For each representation, clearly state the region on which it is valid.

Pf: Observe that $f(z)$ has simple poles at $z = \pm i, 2$.

So we want to consider the regions $R_1 = \{z \in \mathbb{C} : |z| < 1\}$
$R_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$
$R_3 = \{z \in \mathbb{C} : |z| > 2\}$.

First we will rewrite $f(z)$ as follows:

$$f(z) = \frac{1}{(z+i)(z-i)} - \frac{z}{2-z} = \frac{1}{z+i} \cdot \frac{1}{z-i} - \frac{z}{2-z}$$

$$= \frac{-1}{2i} \cdot \frac{1}{z+i} + \frac{1}{2i} \cdot \frac{1}{z-i} - \frac{z}{2-z} = \frac{i}{2} \cdot \frac{1}{z+i} + \frac{i}{2} \cdot \frac{1}{z-i} - \frac{z}{2-z}$$

$$= \frac{i}{2} \cdot \frac{1}{i(1+z/i)} + \frac{i}{2} \cdot \frac{1}{i(1-z/i)} - \frac{z}{2} \cdot \frac{1}{(1-z/2)} = \underline{\frac{1}{2}\left(\frac{1}{1+z/i}\right) + \frac{1}{2}\left(\frac{1}{1-z/i}\right) - \frac{z}{2}\left(\frac{1}{1-z/2}\right)}$$

For $z \in R_1$, we have $\left|\frac{z}{i}\right| = \frac{|z|}{|i|} = |z| < 1$ and $\left|\frac{-z}{i}\right| = \frac{|-z|}{|i|} = |-z| < 1$

moreover, $\left|\frac{z}{2}\right| < \frac{|z|}{2} < 1$.

Thus, we have $f(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{i}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty} z\left(\frac{z}{2}\right)^n$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{i}\right)^n z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-1}{i}\right)^n z^n - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty}\left[\frac{1}{2}\left(\left(\frac{1}{i}\right)^n + \left(\frac{-1}{i}\right)^n\right) + \frac{1}{2^n}\right] z^n \text{ in } R_1.$$ (Just a Taylor series)

Next, for $z \in R_2$, we have

$$f(z) = \frac{1}{1+z^2} - \frac{z}{2-z} = \frac{1}{z^2} \cdot \frac{1}{(1/z^2 + 1)} - \frac{z}{2} \cdot \frac{1}{(1-z/2)} = \frac{1}{z^2} \cdot \frac{1}{1-\left(\frac{-1}{z^2}\right)} - \frac{z}{2} \cdot \frac{1}{1-z/2}$$

and $\left|\frac{-1}{z^2}\right| = \frac{|-1|}{|z^2|} < 1$ and $\left|\frac{z}{2}\right| = \frac{|z|}{|2|} < 1$. Thus, we have

$$f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n - \frac{z}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}} - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n+1} \text{ in } R_2.$$

Lastly, for $z \in R_3$, we have

$$f(z) = \frac{1}{1+z^2} - \frac{z}{2-z} = \frac{1}{z^2(1-(-1/z^2))} - \frac{\cancel{z}}{\cancel{z}} \cdot \frac{1}{(2/z - 1)} = \frac{1}{z^2} \cdot \frac{1}{(1-(-1/z^2))} + \frac{1}{(1-2/z)}$$

and $\left|\frac{1}{z^2}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$, so $f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}} + \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$ in R_3.

□

continued...

② (a) State and prove the maximum principle for holomorphic functions.

Pf: Maximum modulus principle: If f is a nonconstant holomorphic function in a region Ω, then $|f|$ cannot attain a maximum in Ω.

Proof: Suppose that $|f|$ did attain a maximum in Ω at z_0.

Since f is holomorphic, it is an open mapping.

Therefore, if $D \subseteq \Omega$ is a small open disk centered at z_0, then its image $f(D)$ is open and contains $f(z_0)$.

This proves that there are points in $z \in D$ s.t. $|f(z)| > |f(z_0)|$. ↯

This is a contradiction to z_0 being a maximum.

Therefore, we conclude that $|f|$ cannot attain a maximum in Ω. □

(b) Suppose f and g are non-vanishing holomorphic functions on $\mathbb{D}$ which extend continuously to the closed unit disk. If $|f(z)| = |g(z)|$ on the boundary $\{|z|=1\}$, show that $|f(z)| = |g(z)|$ on the whole disk. Hence show there is λ with $|\lambda| = 1$ such that $f(z) = \lambda g(z)$ for all z.

Pf: Since f and g are nonvanishing, let $h(z) = \dfrac{f(z)}{g(z)}$.

Note that h is nonvanishing holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

By maximum modulus principle, $|h|$ attains a maximum on $\overline{\mathbb{D}}$.

On $\overline{\mathbb{D}}$, we have that $|f(z)| = |g(z)|$, so $|h(z)| = \dfrac{|f(z)|}{|g(z)|} = 1$.

Since h is nonvanishing, by the minimum modulus principle, $|h|$ attains a minimum on $\overline{\mathbb{D}}$.

But $|h(z)| = 1$ on $\overline{\mathbb{D}}$ $\forall z \in \mathbb{D}$.

Therefore, $|h(z)| = \left|\dfrac{f(z)}{g(z)}\right| = 1 \Rightarrow |f(z)| = |g(z)|$ on the whole disk.

Note that $|h(z)| = \left|\dfrac{f(z)}{g(z)}\right| = 1 \; \forall z \in \mathbb{D}$.

So $|h|$ attains a maximum in $\mathbb{D}$, so by maximum mod. principle, h is constant, $h = \lambda$ s.t. $|h| = |\lambda| = 1$

So $|h(z)| = \left|\dfrac{f(z)}{g(z)}\right| = 1$ and $h(z) = \dfrac{f(z)}{g(z)} = \lambda \Rightarrow f(z) = \lambda g(z)$ for all z. ($|\lambda| = 1$). □

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(a) Show that $|z^3 - z + 1| > |z|$ when z lies on the imaginary axis in $\mathbb{C}$.

Pf: when z lies on the imaginary axis in $\mathbb{C}$, we have $z = bi$, $b \in \mathbb{R}$.

$$|z^3 - z + 1| = |(bi)^3 - bi + 1| = |-b^3 i - bi + 1| = |-i(b^3+b) + 1| = \sqrt{1^2 + (b^3+b)^2}$$
$$= \sqrt{1 + b^6 + 2b^4 + b^2} > \sqrt{0 + b^2} = |bi| = |z|.$$
$$\hookrightarrow b^6 + 2b^4 + 1 > 0$$
$\square$

(b) Determine the number of roots of $z^3 - z + 1 = ze^z$ that lies in the left half-plane in $\mathbb{C}$ (i.e. the set $z = x + iy$ with $x < 0$).