ary 2016

Does there exist a function f, holomorphic in $\mathbb{C}\setminus\{0\}$ such that $|f(z)| \geq \frac{1}{|z|^{9/10}}$ for all $z \in \mathbb{C}\setminus\{0\}$? Prove your assertion.

Pf. Suppose there does exist such a function f.

Then $|f(z)| \geq \frac{1}{|z|^{9/10}}$ for all $z \in \mathbb{C}\setminus\{0\}$.

Let $g(z) = \frac{1}{f(z)}$. Then $|g(z)| = \left|\frac{1}{f(z)}\right| \leq |z|^{9/10}$ in $\mathbb{C}\setminus\{0\}$.

Since g is holomorphic in $\mathbb{C}\setminus\{0\}$ and g is bounded, by Riemann's removable singularity theorem, we have that g has a removable singularity at $z=0$. (To see this, note $\lim_{z\to 0}|g(z)| = 0$).

Therefore, we can extend g to be an entire function $\tilde{g}$. $(\tilde{g}(0)=0)$

Then $|\tilde{g}(z)| \leq |z|^{9/10}$ for all $z \in \mathbb{C}$.

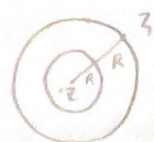
Let $C_{2R}(0)$ be the circle of radius $2R$, $R>0$, centered at $z=0$.

Then for all $z \in B_R(0)$,

$$|g'(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \frac{|g(\zeta)|}{|\zeta-z|^2}\,|d\zeta| \quad \text{since } g \text{ is entire.}$$

$$\leq \frac{1}{2\pi}\int_\gamma \frac{|\zeta|^{9/10}}{|\zeta-z|^2}\,|d\zeta| \quad \text{by assumption.}$$

Note if $z \in B_R(0)$, $\zeta \in C_{2R}(0)$. Then $|\zeta - z| > R$, so



$$|g'(z)| \leq \frac{1}{2\pi}\int_\gamma \frac{|\zeta|^{9/10}}{R^2}\,|d\zeta|$$

$$= \frac{R^{9/10}}{2\pi R^2}\cdot 2\pi R = \frac{R^{9/10}}{R} = \frac{1}{R^{1/10}} \longrightarrow 0 \text{ as } R \to \infty.$$

so $g'(z) \equiv 0$. Therefore, $g(z)$ is constant.

But $\tilde{g}(0)=0$, so $g \equiv 0$. Then $f \equiv \frac{1}{0}$, which is a contradiction. ↯

Therefore, we conclude that such a function f cannot exist. $\square$

Continued...

② Let $Aut(\mathbb{D})$ be the group of holomorphic automorphisms of $\mathbb{D}$ and let Id be the identity map.

(i) For each $b \in \mathbb{D}$, construct a map $\varphi \in Aut(\mathbb{D}) \setminus \{Id\}$ such that b is a fixed point of φ, i.e., $\varphi(b) = b$.

Pf: Let $b \in \mathbb{D}$ s.t. $b \neq 0$.

Let $\varphi_1(z) = \dfrac{b-z}{1-\bar{b}z}$.

Then $\varphi_1(b) = 0$, so we need another automorphism of $\mathbb{D}$ that sends 0 to b.

Let $\varphi_2(z) = \dfrac{b+z}{1+\bar{b}z}$. Then $\varphi_2(0) = b$.

Then $\varphi = \varphi_2 \circ \varphi_1 : \mathbb{D} \to \mathbb{D}$ is an automorphism and $(\varphi_2 \circ \varphi_1)(b) = \varphi_2(0) = b$.

Observe that
$$(\varphi_2 \circ \varphi_1)(z) = \varphi_2\left(\frac{b-z}{1-\bar{b}z}\right)$$
$$= \frac{b + \left(\frac{b-z}{1-\bar{b}z}\right)}{1+\bar{b}\left(\frac{b-z}{1-\bar{b}z}\right)}$$
$$= \frac{\frac{b - b\bar{b}z + b - z}{1-\bar{b}z}}{\frac{1-\bar{b}z+\bar{b}b-\bar{b}z}{1-\bar{b}z}}$$
$$= \frac{2b - b\bar{b}z - z}{1-\bar{b}z} \div \frac{1-2\bar{b}z+\bar{b}b}{1-\bar{b}z}$$
$$= \frac{2b - b\bar{b}z - z}{1-2\bar{b}z+b\bar{b}},$$

so $(\varphi_2 \circ \varphi_1) \neq Id$.

Observe that $\varphi_1(0) = b$ and $\varphi_2(b) \neq 0$, so 0 is not fixed.

So $\varphi_2 \circ \varphi_1$ fixes b but is not the identity.

If $b = 0$, then $\varphi(z) = cz$ for $|c| = 1$, $c \neq 1$.

□

ed...

Does there exist a map $\varphi \in Aut(\mathbb{D})\setminus\{Id\}$ such that φ have two distinct fixed points in $\mathbb{D}$? Prove your assertion.

Pf: Assume there exists $\varphi \in Aut(\mathbb{D})\setminus\{Id\}$ s.t. φ has two distinct fixed points.

First suppose the two fixed points are $\varphi(0)=0$ and $\varphi(a)=a$ $(a\neq 0)$.
Since $\varphi: \mathbb{D}\to\mathbb{D}$ and $\varphi(0)=0$, by Schwarz's lemma we have that $|\varphi(0)|=0$, so φ is a rotation, i.e., $\varphi(z)=\lambda z$ where $|\lambda|=1$

Then $\varphi(a)=\lambda a = a \Rightarrow \lambda = 1$
Therefore, $\varphi(z)=z$ is the identity, but $\varphi\in Aut(\mathbb{D})\setminus\{Id\}$. $\lightning$

Now suppose the two fixed points are $\varphi(a)=a$ and $\varphi(b)=b$ $(a,b\neq 0, a\neq b)$.
Let $\psi(z)=\dfrac{a-z}{1-\bar{a}z}$, which is an automorphism of $\mathbb{D}$ s.t. $\psi=\psi^{-1}$.

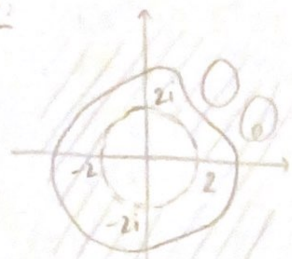
Let $g=\psi^{-1}\circ\varphi\circ\psi$. Then g is also an automorphism of $\mathbb{D}$.
We have
$$\begin{aligned} g(0) &= \psi^{-1}(\varphi(\psi(0)))\\ &= \psi^{-1}(\varphi(a))\\ &= \psi^{-1}(a)\\ &= 0. \end{aligned}$$
We also have
$$\begin{aligned} g(\psi^{-1}(b)) &= \psi^{-1}(\varphi(\psi(\psi^{-1}(b))))\\ &= \psi^{-1}(\varphi(b))\\ &= \psi^{-1}(b). \end{aligned}$$
So we have that g has two fixed points, namely 0 and $\psi^{-1}(b)\neq 0$.
This is our first case, which tells us that $g = Id$.
So
$$\begin{aligned} g = Id_{\mathbb{D}} = \psi^{-1}\circ\varphi\circ\psi \Rightarrow \psi\circ Id_{\mathbb{D}}\circ\psi^{-1} &= \varphi\\ \psi\circ\psi^{-1} &= \varphi\\ Id_{\mathbb{D}} &= \varphi \quad \lightning \end{aligned}$$

This is a contradiction because $\varphi\in Aut(\mathbb{D})\setminus\{Id\}$.

Therefore, we conclude that there does not exist a map $\varphi\in Aut(\mathbb{D})\setminus\{Id\}$ such that φ has two distinct fixed points in $\mathbb{D}$. $\square$

Let $G = \{z \in \mathbb{C} : |z| > 2\}$ and $f(z) = \frac{1}{z^4+1}$. Is there a complex differentiable function on G whose derivative is $f(z)$? Prove your assertion.

Pf:



Recall that f has an analytic antiderivative in $G \Leftrightarrow$ $\int_\gamma f(z)\,dz = 0$ for any closed curve in G.

So it suffices to show that $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in G.

Observe that $f(z) = \frac{1}{z^4+1} = \frac{1}{(z^2+i)(z^2-i)}$ $\Rightarrow$ f has poles at $z = \pm\sqrt{i}, \pm\sqrt{-i}$.

The poles of f are all contained in $\{z \in \mathbb{C} : |z| \le 2\} = \mathbb{C} \setminus G$.

Case 1: Suppose γ is a closed curve in G that does not wind around 0.

Then $\int_\gamma f(z)\,dz = 0$ by Cauchy's theorem since f is analytic in an open nbhd of γ that contains all points bounded by γ.

Case 2: Suppose γ is a closed curve that winds around 0 in G.

WLOG, let $\gamma = re^{it}$, $0 \le t \le 2\pi$, $r > 2$.

The residue theorem says that f just needs to be meromorphic in a region Ω with the singularities not on γ.

Here, $f(z)$ is meromorphic in $\mathbb{C}$ with poles at the 4^{th} roots of unity,

so we just need $\gamma = Re^{it}$, $R > 1$, to use the residue theorem.

By the residue theorem, we have

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \sum_j \operatorname{Res}_f(z_j)$$
$$= 2\pi i \cdot 0$$
$$= 0$$

Therefore, in both cases, $\int_\gamma f(z)\,dz = 0$ for any closed curve in G.

Thus, we conclude that f has an analytic antiderivative in G, i.e., there is a complex differentiable function on G whose derivative is $f(z)$. $\square$

Continued...

(4) Let f be a holomorphic function in $\mathbb{D}$. Suppose that $|f(z)| \leq \frac{1}{1-|z|}$ for all $z \in \mathbb{D}$. Prove that $|f'(z)| \leq \frac{4}{(1-|z|)^2}$ for all $z \in \mathbb{D}$.

Pf: Let $z_0 \in \mathbb{D}$, consider $B_r(z_0)$ where r is s.t. $\overline{B_r(z_0)} \subset \mathbb{D}$

By Cauchy's formula,

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| \leq \frac{1}{2\pi} \int_{\partial B_r(z_0)} \frac{|f(\zeta)|}{|\zeta - z_0|^2} |d\zeta|$$

$$\leq \frac{1}{2\pi} \int_{\partial B_r(z_0)} \frac{1}{|\zeta - z_0|^2} \cdot \frac{1}{(1-|\zeta|)} |d\zeta|$$

we want to bound this so $|\zeta - z_0| > C$

Then $\frac{1}{|\zeta - z_0|^2} < \frac{1}{C^2}$

we are trying to prove something involving $\frac{4}{1-|z_0|^2}$, so if $|\zeta - z_0| = \frac{1-|z_0|}{2}$, then $\frac{1}{|\zeta - z_0|^2} = \frac{4}{(1-|z_0|)^2}$. Then let $r = \frac{1-|z_0|}{2}$.

First, need to verify $B_r(z_0) \subseteq \mathbb{D}$. Note if $z \in B_r(z_0)$, then

$$|z| \leq |z - z_0| + |z_0| < \frac{1-|z_0|}{2} + |z_0| = \frac{1-|z_0|+2|z_0|}{2} = \frac{1+|z_0|}{2}$$

$$\leq \frac{1+1}{2} = 1 \text{ since } z_0 \in \mathbb{D}$$

Hence $z \in \mathbb{D}$, thus, $B_r(z_0) \subseteq \mathbb{D}$.

Now, $\frac{1}{2\pi} \int_{\partial B_r(z_0)} \frac{1}{|\zeta - z_0|^2} \cdot \frac{1}{(1-|\zeta|)} |d\zeta| = \frac{1}{2\pi} \int_{\partial B_r(z_0)} \frac{4}{(1-|z_0|)^2} \cdot \frac{1}{(1-|\zeta|)} |d\zeta|$ since $r = \frac{1-|z_0|}{2}$

$$= \frac{2}{\pi(1-|z_0|)^2} \int_{\partial B_r(z_0)} \frac{1}{1-|\zeta|} |d\zeta|$$

Note $|\zeta| \leq |z_0| + |\zeta - z_0| = |z_0| + \frac{1-|z_0|}{2} = \frac{|z_0|+1}{2}$. Hence, $1 - |\zeta| \geq 1 - \left(\frac{|z_0|+1}{2}\right)$
$= \frac{1-|z_0|}{2}$.

Thus, $\frac{2}{\pi(1-|z_0|)^2} \int_{\partial B_r(z_0)} \frac{1}{1-|\zeta|} |d\zeta| \leq \frac{2}{\pi(1-|z_0|)^2} \int_{\partial B_r(z_0)} \frac{2}{1-|z_0|} |d\zeta| = \frac{4}{\pi(1-|z_0|)^3} \cdot 2\pi \left(\frac{1-|z_0|}{2}\right)$

$$= \frac{4}{(1-|z_0|)^2}$$

Since z_0 was arbitrary, $|f'(z)| \leq \frac{4}{(1-|z|)^2}$ $\forall z \in \mathbb{D}$.

□

area.
How many zeros counting multiplicities does the polynomial $p(z) = z^5 + z^3 + 5z^2 + 2$ have in the region $\{z \in \mathbb{C}; 1 < |z| < 2\}$? Prove your assertion.

Pf. Observe that when $|z| = 2$, we have

$|z^5| = 2^5 = 32$
$|z^3| = 2^3 = 8$
$|5z^2| = 5 \cdot 2^2 = 20$
$|2| = 2$

Let $f(z) = z^5$ and $g(z) = z^3 + 5z^2 + 2$.
On $|z| = 2$, we have $|g(z)| \leq 8 + 20 + 2 = 30 < 32 = |f(z)|$.
By Rouché's theorem, we have that f and $f+g$ have the same number of zeros on $\{z \in \mathbb{C} : |z| < 2\}$.
Since $f(z) = z^5$ has a zero at $z = 0$ w/ mult. 5, we have that $f(z) + g(z) = p(z) = z^5 + z^3 + 5z^2 + 2$ has five zeros in $\{z \in \mathbb{C} : |z| < 2\}$.

Observe that when $|z| = 1$, we have

$|z^5| = 1$
$|z^3| = 1$
$|5z^2| = 5$
$|2| = 2$

Let $f(z) = 5z^2$ and $g(z) = z^5 + z^3 + 2$.
On $|z| = 1$, we have $|g(z)| \leq 1 + 1 + 2 = 4 < 5 = |f(z)|$.
By Rouché's theorem, we have that f and $f+g$ have the same number of zeros on $\{z \in \mathbb{C} : |z| < 1\}$.
Since $f(z) = 5z^2$ has a zero at $z = 0$ w/ mult. 2, we have that $f(z) + g(z) = p(z) = z^5 + z^3 + 5z^2 + 2$ has two zeros in $\{z \in \mathbb{C} : |z| < 1\}$.

Therefore, we conclude that $p(z) = z^5 + z^3 + 5z^2 + 2$ has $5 - 2 = 3$ zeros in $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

$\square$