nuary 2017

(1) Compute $\int_\gamma \frac{6z-7}{z^2-z}\,dz$, where γ is the contour displayed on the right.

Pf: Let $f(z) = \frac{6z-7}{z^2-z} = \frac{6z-7}{z(z-1)}$.

Observe that $f(z)$ has simple poles at $z=0$ and $z=1$.

First we will compute the residues:

$$\operatorname{Res}[f(z); z=1] = \lim_{z\to 1} (z-1)\frac{(6z-7)}{z(z-1)} = \lim_{z\to 1} \frac{6z-7}{z} = -1$$

$$\operatorname{Res}[f(z); z=0] = \lim_{z\to 0} z\cdot\frac{(6z-7)}{z(z-1)} = \lim_{z\to 0} \frac{6z-7}{z-1} = 7$$

From the contour γ, we see that $W_\gamma(0)=1$ and $W_\gamma(1)=2$.

From the generalized residue theorem, we have

$$\begin{aligned}\int_\gamma \frac{6z-7}{z^2-z}\,dz &= 2\pi i \sum_j W_\gamma(z_j)\operatorname{Res}(f(z); z_j)\\ &= 2\pi i\left[1\cdot 7 + 2\cdot(-1)\right]\\ &= 2\pi i\,[7-2]\\ &= 2\pi i\cdot 5\\ &= 10\pi i.\end{aligned}$$

Therefore, $\int_\gamma \frac{6z-7}{z^2-z}\,dz = 10\pi i$.

$\square$

continued...

② Prove that if f is entire, $f(1)=1$, and $|f(z)| \le e^{1/|z|}$ for all $z \neq 0$, then $f(z)=1$ for all $z \in \mathbb{C}$.

Pf: We WTS that f is constant.

Since f is entire, we have that f is continuous on $\overline{\mathbb{D}}$.

By the maximum principle, $|f|$ attains a maximum on $\partial\mathbb{D}$ for all $|z|<1$ $(z \in \mathbb{D})$. Let this maximum be M, so $|f(z)| \le M$ $\forall z \in \overline{\mathbb{D}}$.

If $|z|>1$, then $\frac{1}{|z|}<1$, so $e^{1/|z|} \le e$.

Therefore, we have $|f(z)| \le e^{1/|z|} \le e \Rightarrow |f(z)| \le e$.

Thus, $|f(z)| \le \max\{M, e\}$. So $|f(z)|$ is bounded.

Hence by Liouville's theorem, f is constant.

Since $f(1)=1$, we have that $f(z)=1$ for all $z \in \mathbb{C}$.

$\square$

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(3) Prove that if f is analytic on $\mathbb{D}$, $|f(z)| \le 1$ for all $z \in \mathbb{D}$, and $z_1, \dots, z_n \in \mathbb{D}$ are zeros of f, then $|f(0)| \le \prod_{j=1}^{n} |z_j|$.

(Convention: if z_j is a zero of order k, then z_j may appear in the list $z_1, \dots, z_n$ up to k times.)

Pf Claim: $|f(z)| \le \prod_{j=1}^{n} \left| \frac{z - z_j}{1 - \bar{z}_j z} \right|$.

Let $G(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}$. Notice $G(z)$ is analytic on $\mathbb{D}$ since each $z_j \in \mathbb{D}$, $\frac{1}{\bar{z}_j} \notin \mathbb{D}$ $\left(|z_j| < 1 \Rightarrow \left|\frac{1}{\bar{z}_j}\right| > 1\right)$.

Goal: bound $\frac{f(z)}{G(z)}$.

Let $z \in \partial\mathbb{D}$, $z = e^{i\theta}$. Then

$$\left| \frac{e^{i\theta} - z_j}{1 - \bar{z}_j e^{i\theta}} \right| = \left| \frac{e^{i\theta} - z_j}{e^{i\theta}(e^{-i\theta} - \bar{z}_j)} \right| = \left| \frac{e^{i\theta} - z_j}{e^{-i\theta} - \bar{z}_j} \right| = 1 \quad \text{since } \overline{e^{i\theta} - z_j} = e^{-i\theta} - \bar{z}_j.$$

So $|G(z)| = 1$ on $\partial\mathbb{D}$.

Consider $\frac{f(z)}{G(z)} = \frac{f(z)}{\prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}}$. So $f(z) = \frac{(z - z_j) g(z)}{\prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}}$ (near z_j).

Thus, each z_j is a removable singularity of $\frac{f}{G}$.

So $\frac{f}{G}$ extends to be analytic in $\mathbb{D}$.

Since z_j is a zero of $f(z)$, near z_j, $f(z) = (z - z_j) g(z)$:

$$\frac{f(z)}{G(z)} = \frac{(z - z_j) g(z)}{\left(\frac{z - z_1}{1 - \bar{z}_1 z}\right) \left(\frac{z - z_j}{1 - \bar{z}_j z}\right) \cdots \left(\frac{z - z_n}{1 - \bar{z}_n z}\right)}.$$

Since $\frac{f}{G}$ is analytic in $\mathbb{D}$, $\left|\frac{f(z)}{G(z)}\right| \le \frac{1}{|G(z)|} \le 1$ by maximum modulus principle, since $|G(z)| = 1$ on $\partial\mathbb{D}$.

$\left(\frac{1}{|G(z)|} = 1 \text{ on } \partial\mathbb{D}, \text{ so } \left|\frac{f(z)}{G(z)}\right| \le 1 \text{ in } \mathbb{D}\right)$

Thus, $|f(z)| \le |G(z)|$ $\forall z \in \mathbb{D}$. So we have proven the claim.

Hence, by plugging in $z = 0$, we get $|f(0)| \le \prod_{j=1}^{n} |z_j|$.

$\square$

continued...

④ How many zeros (counting multiplicities) does $p(z) = z^6 + 4z^2 - 5$ have in the annulus $\{1 < |z| < 2\}$?

Pf: Observe that $p(z) = z^6 + 4z^2 - 5 = (z^2-1)(z^4+z^2+5)$, where (z^2-1) has zeros at $z = \pm 1 \notin \{1<|z|<2\}$.

Now we will use Rouché's theorem on z^4+z^2+5 to find how many zeros (counting mult.) it has in $\{1<|z|<2\}$.

On $|z|=2$, we have $|z^4| = 2^4 = 16$

$|z^2| = 2^2 = 4$

$|5| = 5$

Let $f(z) = z^4$ and $g(z) = z^2+5$.
On $|z|=2$, we have $|g(z)| \leq 4+5 = 9 < 16 = |f(z)|$.
By Rouché's theorem, f and $f+g$ have the same number of zeros in $\{z \in \mathbb{C} : |z| < 2\}$. $f(z) = z^4$ has a zero at $z=0$ w/ mult. 4.
Therefore, $f(z)+g(z) = z^4+z^2+5$ has 4 zeros in $\{z \in \mathbb{C} : |z|<2\}$.

On $|z|=1$, we have $|z^4| = 1$

$|z^2| = 1$

$|5| = 5$

Let $f(z) = 5$ and $g(z) = z^4+z^2$.
On $|z|=1$, we have $|g(z)| \leq 1+1 = 2 < 5 = |f(z)|$.
By Rouché's theorem, f and $f+g$ have the same number of zeros in $\{z \in \mathbb{C} : |z|<1\}$. $f(z)=5$ has no zeros.
Therefore, $f(z)+g(z) = z^4+z^2+5$ has 0 zeros in $\{z \in \mathbb{C} : |z|<1\}$.

Thus, $p(z) = z^6+4z^2-5$ has $4-0=4$ zeros in $\{z \in \mathbb{C} : 1<|z|<2\}$.

$\square$

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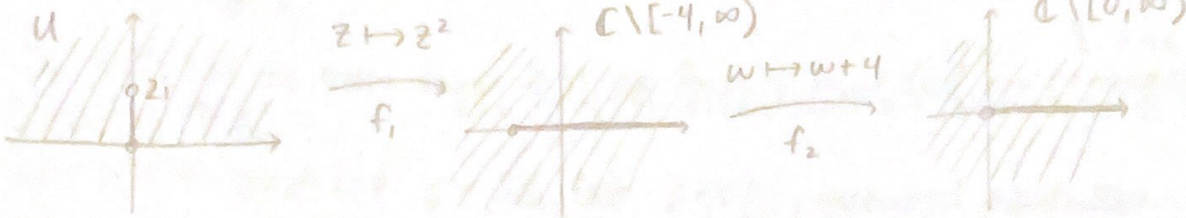
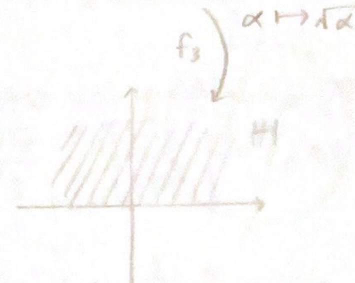
(5) Find a one-to-one analytic map from the domain $U := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \setminus \{bi : 0 \le b \le 2\}$ onto $\mathbb{D}$.

Pf:



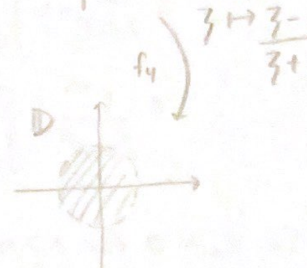


Let $f_1 : U \to \mathbb{C}\setminus[-4,\infty)$ by $f_1(z) = z^2$

$f_2 : \mathbb{C}\setminus[-4,\infty) \to \mathbb{C}\setminus[0,\infty)$ by $f_2(w) = w+4$

$f_3 : \mathbb{C}\setminus[0,\infty) \to \mathbb{H}$ by $f_3(\alpha) = \sqrt{\alpha}$

$f_4 : \mathbb{H} \to \mathbb{D}$ by $f_4(\zeta) = \frac{\zeta - i}{\zeta + i}$.



Let $f : U \to \mathbb{D}$ by $f(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z)$

f is 1-1 and conformal b/c comp of 1-1 is 1-1 and comp. of conformal is conformal.

□

continued...

(6) Let m be a positive integer. Prove that if f is an entire function and $|f(z)| \leq |z|^m$ for all $z \in \mathbb{C}$, then f is a polynomial of degree at most m.

Pf: Let $m \in \mathbb{Z}^+$.

Since f is entire, we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

The coefficients are $a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$.

By bounding the coefficients, we get

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |dz| \leq \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{r^{n+1}} |dz|$$

$$\leq \frac{1}{2\pi r^{n+1}} \int_{|z|=r} |f(z)||dz| \leq \frac{1}{2\pi r^{n+1}} \int_{|z|=r} |z|^m |dz| \leq \frac{1}{2\pi r^{n+1}} \int_{|z|=r} r^m |dz|$$

$$\leq \frac{r^m}{2\pi r^{n+1}} \int_{|z|=r} |dz| = \frac{r^m}{2\pi r^{n+1}} \cdot 2\pi r = \frac{1}{r^{n-m}} \to 0 \text{ as } r \to \infty \text{ for } n > m.$$

So $a_n = 0$ for $n > m$.

Therefore, f is a polynomial of degree at most m.

□

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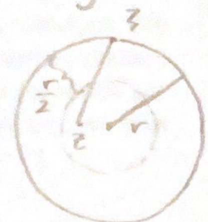
⑦ Let $\mathcal{F}$ be a family of analytic maps defined on $\mathbb{D}$. Prove that if $M_r := \sup_{f \in \mathcal{F}} \int_{|z|=r} |f(z)||dz| < \infty$ for all $0<r<1$, then $\mathcal{F}$ is a normal family.

Pf: It suffices to show that $\mathcal{F}$ is uniformly bounded on compact subsets of $\mathbb{D}$.

Let $f \in \mathcal{F}$ and let $\gamma = \partial B_r(0)$, $0<r<1$, oriented positively.

By Cauchy's formula, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$.



Then $|f(z)| \le \frac{1}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|\zeta - z|}|d\zeta| \quad \forall z \in B_{\frac{r}{2}}(0)$.

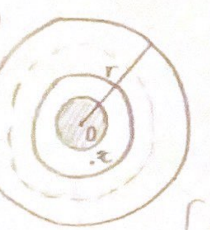
Then $|\zeta - z| > \frac{r}{2}$.

So $|f(z)| \le \frac{1}{2\pi\left(\frac{r}{2}\right)}\int_{\gamma \to |z|=r} |f(\zeta)||d\zeta| \quad \forall z \in B_{\frac{r}{2}}(0)$

$\le \frac{1}{\pi r} \cdot M_r$ (by assumption)

$\mathcal{F}$ is uniformly bounded in $B_{\frac{r}{2}}(0)$, $0<r<1$.

$\mathbb{D}$



$(B_{\frac{1}{2}}(0))$ for $r \ge \frac{1}{2}$

Let $\gamma = \partial B_{r+\frac{1-r}{2}}(0) = \partial B_{\frac{r+1}{2}}(0)$.

By Cauchy's theorem,

$|f(z)| \le \frac{1}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|\zeta - z|}|d\zeta| \quad \forall z \in B_{\frac{r+1}{2}}(0)$

$\left[|\zeta - z| > \frac{1-r}{2}\right]$

$\le \frac{1}{2\pi\left(\frac{1-r}{2}\right)}\int_\gamma |f(\zeta)||d\zeta|$

$\le \frac{1}{\pi(1-r)} \cdot M_r \quad \forall z \in B_{\frac{r+1}{2}}(0)$.

$\mathcal{F}$ is uniformly bounded in $B_r(0)$, $0<r<1$.

Let K be a compact subset of $\mathbb{D}$. Then we can take r large enough so $K \subseteq B_r(0) \subseteq \mathbb{D}$.

Then by above, $\mathcal{F}$ is uniformly bounded on K.

Hence, $\mathcal{F}$ is a normal family.

□