ary 2018

Let $v: \mathbb{C}\setminus[0,\infty) \to (0, 2\pi)$ be the function $v(x+iy) = \arg(x+iy)$, $x, y \in \mathbb{R}$.
($\mathbb{C}\setminus[0,\infty)$ is the complement in $\mathbb{C}$ of $\{x+0i : x \geq 0\}$.)

(a) Compute $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

Pf: We have that for $x+iy \in \mathbb{C}\setminus[0,\infty)$

$$v(x+iy) = \operatorname{arc}(x+iy) = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{1}{x+\frac{y^2}{x}} = \frac{x}{x^2+y^2}$$

Therefore, $\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$.

$\square$

(b) Determine whether v is harmonic

Pf: Recall that v is harmonic if $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Observe that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} = 0$.

Therefore, v is harmonic.

$\square$

continued...

(2) Suppose that f is an entire function satisfying $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a polynomial.

Pf: Since f is an entire function, we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Since $\lim_{|z|\to\infty} |f(z)| = \infty$, we have that $\lim_{|z|\to 0} \left|f\left(\frac{1}{z}\right)\right| = \infty$.

So we have that $f\left(\frac{1}{z}\right)$ has a pole.

We can write $f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$, but since $f\left(\frac{1}{z}\right)$ has a pole, we know that after some $n=k$, the $a_n = 0$.

So we have $f\left(\frac{1}{z}\right) = \sum_{n=0}^{k} \frac{a_n}{z^n}$.

Therefore, we have that $f(z) = \sum_{n=0}^{k} a_n z^n$.

Thus, f is a polynomial. $\square$

...ued...

(a) Prove that $(z^2-1)^{-1}$ has an analytic square root on the domain $\mathbb{C}\setminus[-1,1]$.
($\mathbb{C}\setminus[-1,1]$ is the complement in $\mathbb{C}$ of the line segment from -1 to 1.)

Pf: $(z^2-1)^{-1} = \frac{1}{z^2-1}$

Observe that $\log\left(\frac{1}{z^2-1}\right)$ is well-defined on $\mathbb{C}$ minus a branch cut so $\mathbb{C}\setminus(-\infty,0]$

$\frac{1}{z^2-1} \in \mathbb{R}$ if $z \in \mathbb{R}$

$\frac{1}{z^2-1} \leq 0$ whenever $z \in [-1,1]$

$\Leftrightarrow$

$\frac{1}{z^2-1} : \mathbb{C}\setminus[-1,1] \longrightarrow \mathbb{C}\setminus\{(x,0) : x \leq 0\}$

so we have that $\log\left(\frac{1}{z^2-1}\right)$ is holomorphic.

Therefore, $e^{\frac{1}{2}\log\left(\frac{1}{z^2-1}\right)}$ is the analytic square root of (z^2-1) on $\mathbb{C}\setminus[-1,1]$.

$\square$

(b) Find the Laurent expansion of an analytic square root from part (a) on the domain $\{z : |z|>1\}$, centered about $z=0$.

continued...

(4) Let $f: \mathbb{D} \to \mathbb{C}$ be an analytic function. Assume that $|f(z)| \le 1$ for all $z \in \mathbb{D}$, and that f has a zero of order $m \ge 1$ at the origin. Show that $|f(z)| \le |z|^m$.

Pf: Since f has a zero of order $m \ge 1$, we have that

$\frac{f(z)}{z^m} = g(z)$ is holomorphic $(g(0) \neq 0)$.

Then $f(z) = z^m g(z)$.

We are given that $|f(z)| = |z^m g(z)| \le 1$

$$\Rightarrow |z^m||g(z)| \le 1$$
$$|g(z)| \le \frac{1}{|z|^m}$$

Observe that $g: \mathbb{D} \to \mathbb{C}$ is holomorphic. Let $0<r<1$ and $B(0,r) \subseteq \mathbb{D}$. By the maximum modulus principle, $|g(z)|$ attains a maximum on $\overline{B(0,r)}$

So $|g(z)| \le \frac{1}{|z|^m} = \frac{1}{r^m}$.

Letting $r \to 1$, we see that $|g(z)| \le \frac{1}{r^m} = \frac{1}{1} = 1$.

Therefore, $|g(z)| \le 1$.

Thus, we have $\left|\frac{f(z)}{z^m}\right| = |g(z)| \le 1$

$$\Rightarrow \frac{|f(z)|}{|z^m|} \le 1$$

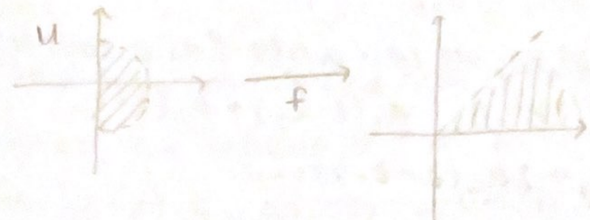
$$\Rightarrow |f(z)| \le |z|^m.$$

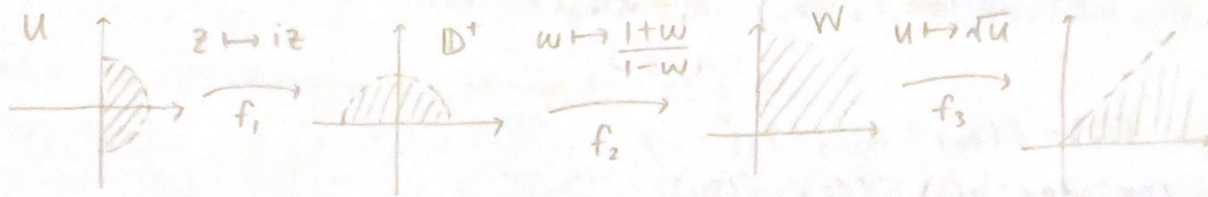
$\square$

...nued...

Find a conformal mapping from $\mathbb{D} \cap \{z : \operatorname{Re} z > 0\}$ onto the wedge $\{z \in \mathbb{C} : 0 < \operatorname{Im}(z) < \operatorname{Re}(z)\}$.

Pf: We want $f : \mathbb{D} \cap \{z : \operatorname{Re}(z) > 0\} \to \{z \in \mathbb{C} : 0 < \operatorname{Im}(z) < \operatorname{Re}(z)\}$





Let $f_1 : U \to \mathbb{D}^+$ by $f_1(z) = iz$,

$f_2 : \mathbb{D}^+ \to W$ by $f_2(w) = \dfrac{1+w}{1-w}$,

$f_3 : W \to V$ by $f_3(u) = \sqrt{u}$.

Let $f : U \to V$ by $f(z) = (f_3 \circ f_2 \circ f_1)(z)$.

Observe that f is conformal b/c the composition of conformal is conformal.

$\square$

continued...

⑥ Suppose that f is analytic and one-to-one on a domain U. Prove that does not have zeros on U.

Pf: Assume by way of contradiction that there exists $z_0 \in U$ such that $f'(z_0) = 0$.

Since f is analytic at z_0, we can write f as a convergent power series centered at z_0: $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

Note that $f'(z) = a_1 + 2a_2(z-z_0) + \ldots$

So by assumption, $f'(z_0) = a_1 + 2a_2(z_0 - z_0) + \ldots$
$= a_1 = 0$.

Then $f(z) = f(z_0) + a_2(z-z_0)^2 + \ldots$

So consider $g(z) = f(z) - f(z_0)$
$= a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots$

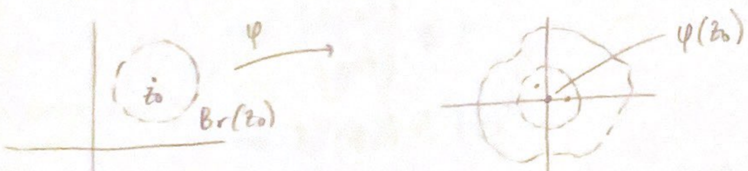
Then there exists $k \in \mathbb{Z}$, $k \geq 2$ such that $g(z) = (z-z_0)^k h(z)$, where h is analytic and nonzero in a nbhd of z_0.

Since h is nonzero near z_0, we can define a branch of $\log h(z)$ which is analytic in $B_r(z_0)$ for r sufficiently small.

Let $H(z) = h(z)^{1/k} = e^{\frac{1}{k}\log h(z)}$.

Then $[H(z)]^k = h(z)$. Hence, $g(z) = [(z-z_0)H(z)]^k$.

Note $\varphi(z) = (z-z_0)H(z)$ is analytic near z_0 so by the open mapping theorem there exists $\delta > 0$ such that $B_{2\delta}(\varphi(z_0)) \subset \varphi(B_r(z_0))$.



Thus, there exists $z_1, z_2 \in B_r(z_0)$, such that $\varphi(z_1) = \varphi(z_0) + \delta$
$= 0 + \delta = \delta$

and $\varphi(z_2) = \varphi(z_0) + \delta e^{\frac{i2\pi}{k}} = \delta e^{\frac{i2\pi}{k}}$, where $\delta \neq \delta e^{i2\pi/k}$ since $k \geq 2$.

Thus, $z_1 \neq z_2$. Then $\left.\begin{array}{l} g(z_1) = (\varphi(z_1))^k = \delta^k \\ g(z_2) = (\varphi(z_2))^k = \delta^k e^{2\pi i} = \delta^k \end{array}\right\}$ So $g(z_1) = g(z_2)$

$f(z_1) - f(z_0) = f(z_2) - f(z_0) \Rightarrow f(z_1) = f(z_2)$ ↯ contradicting the fact that f is injective in U.

Hence, f' has no zeros in U.

□

...ued...

Let $F(z) = \frac{\pi}{z^4} \cdot \frac{\cos(\pi z)}{\sin(\pi z)}$.

(a) Find all poles of F and compute the residue of F at each pole.

Pf: F has simple poles at $z = \pm 1, \pm 2, \ldots,$ and a pole at $z = 0$ with order 5.

We will use the residue theorem to find each residue:

First, we will compute the residue of F at $z = n$ $(n = \pm 1, \pm 2, \ldots)$:

$$\operatorname{Res}[F(z); z=n] = \lim_{z \to n} (z-n) \cdot \frac{\pi}{z^4} \cdot \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{\pi \cos(n\pi)}{n^4} \lim_{z \to n} \frac{(z-n)}{\sin(\pi z)} = \frac{1}{n^4 \pi}.$$

Now, we will compute the residue of F at $z = 0$:

$$\frac{\pi \cos(\pi z)}{z^4 \sin(\pi z)} = \frac{\pi}{z^4} \left(\frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots}{\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \cdots} \right)$$

$$= \frac{1}{z^5} \left(\frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots}{1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \cdots} \right)$$