ary 2019

How many solutions (counted with multiplicity) does the equation $z^6+5z^3+1=0$ have in the unit disk $\mathbb{D}$?

Pf: On $\partial\mathbb{D}$ ($|z|=1$), we have $|z^6|=1$

$|5z^3|=5$

$|1|=1$

Let $f(z)=5z^3$ and $g(z)=z^6+1$.

Then on $\partial\mathbb{D}$, we have that

$|g(z)| \le 1+1=2<3=|f(z)|$.

Therefore, by Rouché's theorem, f and $f+g$ have the same number of zeros in $\mathbb{D}$.

$f(z)=5z^3$ has a zero at $z=0$ with multiplicity 3.

Thus, $f(z)+g(z)=z^6+5z^3+1=0$ has three zeros in the unit disk $\mathbb{D}$. $\square$

Continued...

② Let f be a holomorphic map of the unit disk $\mathbb{D}$ into itself. Suppose f is not the identity map. Can f have two or more fixed points? Prove your assertion. (Recall $w \in \mathbb{D}$ is a fixed point of f if $f(w) = w$.)

Pf: Suppose f has two fixed points.

Case 1: Assume $f(0) = 0$ and $f(b) = b$, $b \neq 0$.

By Schwarz lemma, $|f(z)| \leq |z|$.

Notice that for $z = b$, $|f(b)| = |b|$.

Thus, $f(z) = cz$, $|c| = 1$ (f is a rotation).

So $f(b) = b = bc \Rightarrow c = 1$. Thus, $f(z) = z$ $\lightning$ b/c f is not the id. map.

Case 2: Assume $f(a) = a$ and $f(b) = b$, $a, b \neq 0$.

Define $\varphi(z) = \dfrac{a - z}{1 - \bar{a}z}$, which is an automorphism of $\mathbb{D}$ with $\varphi = \varphi^{-1}$.

$\varphi(0) = a$ and $\varphi^{-1}(a) = 0$.

Let $g(z) = (\varphi^{-1} \circ f \circ \varphi)(z)$. Then g is also an aut. of $\mathbb{D}$.

We have that $g(0) = \varphi^{-1}(f(\varphi(0))) = \varphi^{-1}(f(a)) = \varphi^{-1}(a) = 0$.

Moreover, $g(\varphi^{-1}(b)) = \varphi^{-1}(f(\varphi(\varphi^{-1}(b)))) = \varphi^{-1}(f(b)) = \varphi^{-1}(b)$.

This implies that g has two fixed points, 0 and $\varphi^{-1}(b)$.

By the first case, since 0 is one fixed pt and $\varphi^{-1}(b) \neq 0$ is the other, we get that g is the identity map on $\mathbb{D}$.

Thus, $id_{\mathbb{D}} = \varphi^{-1} \circ f \circ \varphi \Rightarrow \varphi \circ id_{\mathbb{D}} \circ \varphi^{-1} = f \Rightarrow \varphi \circ \varphi^{-1} = f \Rightarrow id_{\mathbb{D}} = f$. $\lightning$

Contradiction b/c f is not the identity map.

Therefore, f cannot have two or more fixed points.

...nued..

Prove or disprove that there exists a holomorphic function $f(z)$ defined on the punctured disk $\mathbb{D}\setminus\{0\}$ such that

$$\lim_{z\to 0} z f(z) = 0 \quad \text{and} \quad \lim_{z\to 0} |f(z)| = \infty.$$

Pf. By Riemann's removable singularity theorem,

$\lim_{z\to 0} zf(z) = 0$ implies that $z=0$ is a removable singularity of $f(z)$.

But $\lim_{z\to 0} |f(z)| = \infty$ implies that $z=0$ is a pole.

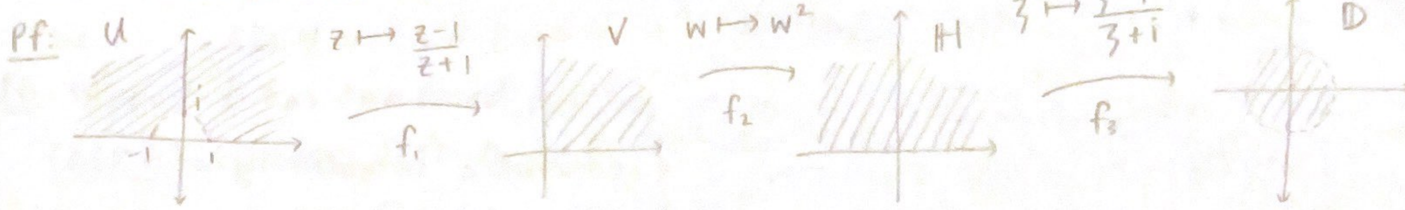
$z=0$ cannot be both a removable sing. and pole.

Therefore, the statement is false. $\square$

Continued...

④ Find a one-to-one conformal map from $U = \{z \in \mathbb{C} : |z| > 1 \text{ and } \operatorname{Im}(z) > 0\}$ onto the unit disk $\mathbb{D}$.

Pf:



Let $f_1 : U \to V$ by $f_1(z) = \frac{z-1}{z+1}$

$f_2 : V \to \mathbb{H}$ by $f(w) = w^2$

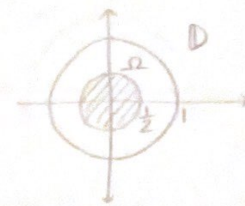
$f_3 : \mathbb{H} \to \mathbb{D}$ by $f(\zeta) = \frac{\zeta - i}{\zeta + i}$

Let $f : U \to \mathbb{D}$ by $f(z) = (f_3 \circ f_2 \circ f_1)(z)$.

f is conformal b/c the comp. of conformal maps is conformal, and f is 1-1 b/c the comp. of 1-1 maps is 1-1. □

ued...

Suppose f is a non-constant holomorphic function on $\mathbb{D}$. Suppose $|f|$ is constant on the circle $|z| = \frac{1}{2}$. Show that f has at least one zero in $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$.



Pf: By way of contradiction, assume $f(z) \neq 0 \ \forall z \in \Omega$.

Since f is holomorphic in $\mathbb{D}$, it is continuous on $\overline{\Omega}$, so by the minimum modulus principle, $|f(z)|$ attains a minimum on $\partial\Omega$.

Also by the maximum modulus principle, $|f(z)|$ attains a maximum on $\partial\Omega$.

So $\exists z_1 \in \partial\Omega$ s.t. $|f(z_1)| = \min_{z \in \overline{\Omega}} |f(z)|$ and

$\exists z_2 \in \partial\Omega$ s.t. $|f(z_2)| = \max_{z \in \overline{\Omega}} |f(z)|$.

On $\partial\Omega$, $|f(z)| = c$, where c is some constant.

So $|f(z_1)| = c = |f(z_2)|$.

Thus, $|f(z)| = c \ \forall z \in \overline{\Omega}$.

By another application of maximum modulus, we get that f is constant in Ω. Since $|f|$ is constant in Ω, it attains a maximum in Ω. Hence by max. principle, f is constant in Ω.

Since Ω is an open subset of $\mathbb{D}$, f must be constant in all of $\mathbb{D}$, by the identity theorem.

This is a contradiction since f is non-constant on $\mathbb{D}$.

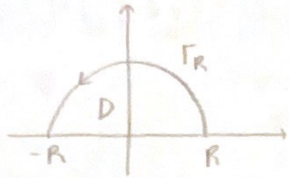
Therefore, f has at least one zero in $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$.

$\square$

Continued...

⑥ Let a be a positive real number. Compute $\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx$.

Pf:



Note that $e^{iaz} = \cos(az) + i\sin(az)$.

Let $f(z) = \frac{e^{iaz}}{(1+z^2)^2}$. The function $f(z)$ has double poles at $z = \pm i$, but only $z = i \in D$.

The residue at $z = i$ is
$$\operatorname{Res}[f(z); z=i] = \lim_{z \to i} \left[\frac{d}{dz} (z-i)^2 \frac{e^{iaz}}{(z+i)^2(z-i)^2}\right]$$
$$= \lim_{z\to i}\left[\frac{d}{dz}\frac{e^{iaz}}{(z+i)^2}\right]$$
$$= \lim_{z\to i}\left[\frac{(z+i)^2 iae^{iaz} - e^{iaz}2(z+i)}{(z+i)^4}\right]$$
$$= \frac{(2i)^2 iae^{-a} - e^{-a}2(2i)}{(2i)^4}$$
$$= \frac{-4iae^{-a} - 4ie^{-a}}{16} = \frac{-ie^{-a}(a+1)}{4}$$

By the residue theorem, we have
$$\int_{\partial D} f(z)\,dz = 2\pi i \cdot \sum_j \operatorname{Res}[f(z): z_j]$$
$$= 2\pi i\left(\frac{-ie^{-a}(a+1)}{4}\right) = \frac{\pi e^{-a}(a+1)}{2}$$

$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{(1+z^2)^2}\,dz\right| \le \int_{\Gamma_R} \frac{|e^{iaz}|}{|(1+z^2)^2|}|dz| \le \int_{\Gamma_R} \frac{1}{|1+z^2|^2}|dz| \le \int_{|z|=R} \frac{1}{(|z|^2-1)^2}|dz|$$

$(|e^{iaz}| \le 1$ and $|(1+z^2)^2| = |1+z^2|^2 \ge (|z|^2-1)^2)$
$$\le \int_{|z|=R} \frac{1}{(R^2-1)^2}|dz|$$
$$= \frac{1}{(R^2-1)^2}\int_{\Gamma_R}|dz| = \frac{\pi R}{(R^2-1)^2}$$
$$\sim \frac{1}{R^3} \to 0 \text{ as } R \to \infty.$$

So we have
$$\int_{\partial D} f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \underbrace{\int_{\Gamma_R} f(z)\,dz}_{\to 0}.$$

$$\int_{-R}^{R} \frac{e^{iax}}{(1+x^2)^2}\,dx = \int_{-R}^{R}\frac{\cos(ax)+i\sin(ax)}{(1+x^2)^2}\,dx = \int_{-R}^{R}\frac{\cos(ax)}{(1+x^2)^2}\,dx + i\int_{-R}^{R}\frac{\sin(ax)}{(1+x^2)^2}\,dx$$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos(ax)}{(1+x^2)^2}\,dx = \frac{\pi e^{-a}(a+1)}{2}$$

Therefore, $\int_{-\infty}^{\infty}\frac{\cos(ax)}{(1+x^2)^2}\,dx = \frac{\pi e^{-a}(a+1)}{2}$. □

...ued...

Is there a one-to-one conformal map from the punctured disk $\mathbb{D} \setminus \{0\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$? Prove your assertion.

Pf: Assume that such a 1-1 conformal map f exists, i.e.,

$f : \mathbb{D} \setminus \{0\} \to A$ is 1-1 analytic.

Then $f^{-1} : A \to \mathbb{D} \setminus \{0\}$ is also 1-1 and analytic.

Since f is bounded near 0, 0 is a removable singularity of f.

Therefore, f extends to be analytic on all of $\mathbb{D}$.

By the open mapping theorem, for some $r > 0$ small,

$f(B_r(0)) \subseteq f(\mathbb{D})$, so

Let $f(0) = w \in Int(A)$.

Since f is 1-1 and onto, $\exists z_1$ s.t. $f(z_1) = w$, $z_1 \neq 0$.

Since $\mathbb{C}$ is Hausdorff, $\exists$ open nbhds U, V of 0 and z_1, resp., s.t. $U \cap V = \emptyset$.

Since f is open, $f(U)$ and $f(V)$ are open.

So $f(U) \cap f(V)$ is open with $w \in f(U) \cap f(V)$.

Therefore, $\exists$ a nbhd B of w s.t. $w \in B \subseteq f(U) \cap f(V)$.

Thus, $\exists w' \in B$, $w' \neq w$ s.t. $w' \in f(U) \cap f(V)$

$$\left.\begin{array}{l} w' \in f(U) \Rightarrow \exists x_1 \in U \text{ s.t. } f(x_1) = w' \\ w' \in f(V) \Rightarrow \exists x_2 \in V \text{ s.t. } f(x_2) = w' \end{array}\right\} x_1 \neq x_2 \text{ b/c } U \cap V = \emptyset$$

Therefore, f is not 1-1. $\lightning$ Contradiction since f is 1-1.

Thus, we conclude that there does not exist such a conformal map from $\mathbb{D} \setminus \{0\}$ onto $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

$\square$