...uary 2020

1) Let f be an entire function. Assume $\int_0^{2\pi} |f(re^{i\theta})|\,d\theta \le r^{20}$ for all $r \ge 100$.

Show that f is a polynomial with degree ≤ 20.

Pf: Since f is entire, we can write $f(z)$ as a convergent power series centered at $z=0$: $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Fix $r \ge 100$.

By Cauchy's formula, $a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz$.

Thus,

$$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz\right| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}}\,|dz|$$

$$\le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z)|}{r^{n+1}}\,|dz|$$

$$\le \frac{1}{2\pi r^{n+1}}\int_{|z|=r} |f(z)|\,|dz|$$

[Let $z = re^{i\theta}$, $dz = ire^{i\theta}d\theta$] $\le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} |f(re^{i\theta})|\,|ire^{i\theta}d\theta|$

$$\le \frac{1}{2\pi r^{n}}\int_0^{2\pi} |f(re^{i\theta})|\,|d\theta|$$

$$\le \frac{1}{2\pi r^{n}} \cdot r^{20}$$

$$= \frac{1}{2\pi r^{n-20}} \longrightarrow 0 \text{ as } r \to \infty \text{ for } n > 20.$$

since this holds for all $r \ge 100$, letting $r \to \infty$ we see

So $a_n = 0$ for $n > 20$.

Therefore, f is a polynomial with degree ≤ 20.

$\square$

continued...

② How many zeros counting multiplicities does the polynomial $p(z) = 2z^5 + z^4 + 7z^2 + 2$ have in the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$? Prove your assertion.

Pf: On $|z| = 2$, we have that
$$|2z^5| = 2^6 = 64$$
$$|z^4| = 2^4 = 16$$
$$|7z^2| = 28$$
$$|2| = 2$$

Let $f(z) = 2z^5$ and $g(z) = z^4 + 7z^2 + 2$.
Then on $|z| = 2$, we have $|g(z)| \leq 16 + 28 + 2 = 46 < 64 = |f(z)|$.
Therefore, by Rouché's theorem, $f(z)$ and $f(z) + g(z) = p(z)$ have the same number of zeros in $\{z \in \mathbb{C} : |z| < 2\}$.
The function $f(z) = 2z^5$ has one zero at $z = 0$ with multiplicity 5.

On $|z| = 1$, we have that
$$|2z^5| = 2$$
$$|z^4| = 1$$
$$|7z^2| = 7$$
$$|2| = 2$$

Let $f(z) = 7z^2$ and $g(z) = 2z^5 + z^4 + 2$.
Then on $|z| = 1$, we have $|g(z)| \leq 2 + 1 + 2 = 5 < 7 = |f(z)|$.
Therefore, by Rouché's theorem, $f(z)$ and $f(z) + g(z) = p(z)$ have the same number of zeros in $\{z \in \mathbb{C} : |z| < 1\}$.
The function $f(z) = 7z^2$ has one zero at $z = 0$ with multiplicity 2.

Therefore, the function $p(z) = 2z^5 + z^4 + 7z^2 + 2$ has $5 - 2 = 3$ zeros counting multiplicities in the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$. $\square$

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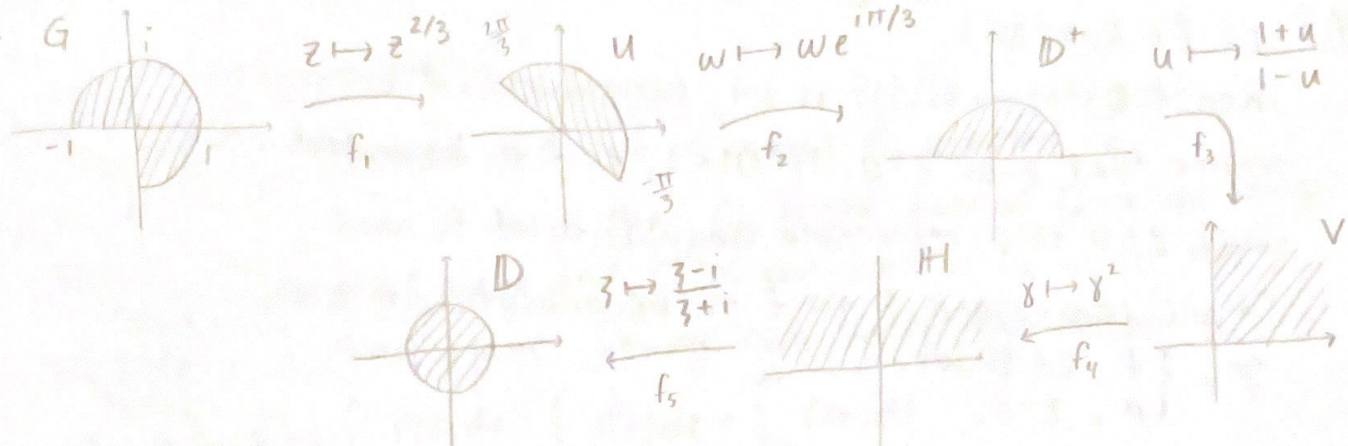
Let $G = \{re^{i\theta} : 0 < r < 1, -\pi/2 < \theta < \pi\}$. Explicitly describe a one-to-one conformal map of G onto the unit disk $\mathbb{D}$, by using an explicit formula.

Pf.



Let $f_1 : G \to U$ by $f_1(z) = z^{2/3}$, then by taking a branch cut on the negative real axis, f_1 is analytic and maps G onto $\{z : |z| < 1, -\frac{\pi}{3} < \arg(z) < \frac{2\pi}{3}\}$.

$f_2 : U \to \mathbb{D}^+$ by $f_2(w) = we^{i\pi/3}$,

$f_3 : \mathbb{D}^+ \to V$ by $f_3(u) = \frac{1+u}{1-u}$,

$f_4 : V \to \mathbb{H}$ by $f_4(\gamma) = \gamma^2$,

$f_5 : \mathbb{H} \to \mathbb{D}$ by $f_5(\zeta) = \frac{\zeta - i}{\zeta + i}$.

Let $f : G \to \mathbb{D}$ by $f(z) = (f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(z)$.

This f is a conformal map and one-to-one since the composition of conformal maps is conformal and the composition of one-to-one maps is one-to-one. □

* Use words to explain pictures.

* Need branch cut for $z \mapsto z^\alpha$ if $\alpha < 1$.

continued...

(4) Let $\mathbb{D}^* = \mathbb{D}\setminus\{0\} = \{0<|z|<1\}$. Find the holomorphic automorphism group $Aut(\mathbb{D}^*)$ (i.e., find all biholomorphic maps from $\mathbb{D}^*$ onto itself). Prove your assertion.

Pf: Let $f \in Aut(\mathbb{D}^*)$.

Then $f: \mathbb{D}\setminus\{0\} \to \mathbb{D}\setminus\{0\}$ is 1-1, surjective, and holomorphic.

Notice that near $z=0$, $|f(z)|<1$, so f is bounded near 0.

Thus, $z=0$ is a removable singularity of f.

So we can extend f to $\tilde{f}$ to be analytic at $z=0$.

$$\tilde{f} = \begin{cases} f, & z \in \mathbb{D}\setminus\{0\}, \\ a, & z=0. \end{cases}$$

By the open mapping theorem, if U is an open nbhd of 0 s.t. $U \subseteq D$, then $\tilde{f}(U)$ must be open. Hence, we must have $\tilde{f}(0) \in Int(D)$.

Assume $\tilde{f}(0) = a \in Int(D)$.

Since f is bijective, $\exists\ b \in \mathbb{D}\setminus\{0\}$ such that $f(b)=a$.

Then let U and V be disjoint open nbhds of 0 and b (resp.) in $\mathbb{D}$.

We know since $\tilde{f}$ is an open map, $\tilde{f}(U)$ and $\tilde{f}(V)$ are open.

Therefore, $\tilde{f}(U) \cap \tilde{f}(V)$ is open. (finite intersection of open sets)

Since $0 \in \tilde{f}(U) \cap \tilde{f}(V)$, there exists an open nbhd N of 0 s.t. $N \subseteq \tilde{f}(U) \cap \tilde{f}(V)$, i.e., $\exists\ z \in N$ s.t. $z \neq 0$ and $z \in \tilde{f}(U)$ and $z \in \tilde{f}(V)$.

Since $U \cap V = \emptyset$, this implies there exist $z_1 \in U$, $z_2 \in V$ s.t. $\tilde{f}(z_1) = \tilde{f}(z_2) = z$. ↯ contradicting the fact that f is injective.

Hence our extension must map 0 to 0 i.e., $\tilde{f}(0)=0$.

Thus, our extension is still an automorphism of $\mathbb{D}$ so it must have the form $\tilde{f}(z) = c\left(\frac{z-a}{1-\bar{a}z}\right)$ $|a|<1$, $|c|=1$, but $\tilde{f}(0)=0$ implies

$$\tilde{f}(0) = c\left(\frac{0-a}{1-\bar{a}\cdot 0}\right) = c(-a), \text{ so } a=0.$$

Thus, $\tilde{f}(z) = c\left(\frac{z-0}{1-\bar{0}\cdot z}\right) = cz$ is a rotation.

Since $\tilde{f} = f$ in $\mathbb{D}\setminus\{0\}$, we have $f(z)=cz$ is a rotation.

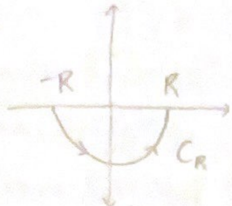
Hence, $Aut(\mathbb{D}^*) = \{f(z) : f(z) = cz, |c|=1\}$.

□

ued...

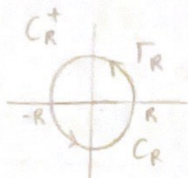
Let C_R be the lower semi-circle of radius $R>0$, i.e., $C_R = \{Re^{i\theta};\ \pi \le \theta \le 2\pi\}$ with positive direction. Compute the limit $\lim_{R\to+\infty} \int_{C_R} \frac{e^{iz}}{z}\,dz$.

Pf:



Let $f(z) = \frac{e^{iz}}{z}$.

Consider the circle of radius R, denoted Γ_R.



Then we have that C_R is the part of Γ_R in the lower half plane and C_R^+ is the part of Γ_R in the upper half plane. So we can write the following:

$$\int_{\Gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{C_R^+} f(z)\,dz.$$

By the residue theorem, we have that $\int_{\Gamma_R} f(z)\,dz = 2\pi i \sum_j^n \operatorname{Res}[f(z); z_j]$.

Notice that $f(z)$ has a simple pole at $z=0$, so

$$\operatorname{Res}\left[\frac{e^{iz}}{z}; z=0\right] = \lim_{z\to 0} (z)\,\frac{e^{iz}}{z} = \lim_{z\to 0} e^{iz} = 1.$$

So $\int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot 1 = 2\pi i$.

$$\left|\int_{C_R^+} \frac{e^{iz}}{z}\,dz\right| \le \int_{C_R^+} \frac{|e^{iz}|}{|z|}|dz| \le \int_{C_R^+} \frac{|e^{iz}|}{R}|dz| = \frac{1}{R}\int_{C_R^+} |e^{iz}||dz| \underbrace{<}_{\text{By Jordan's lemma}} \frac{1}{R}\cdot \pi \to 0 \text{ as } R\to\infty.$$

Therefore, we have $\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = \lim_{R\to\infty}\int_{C_R^+} f(z)\,dz + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$

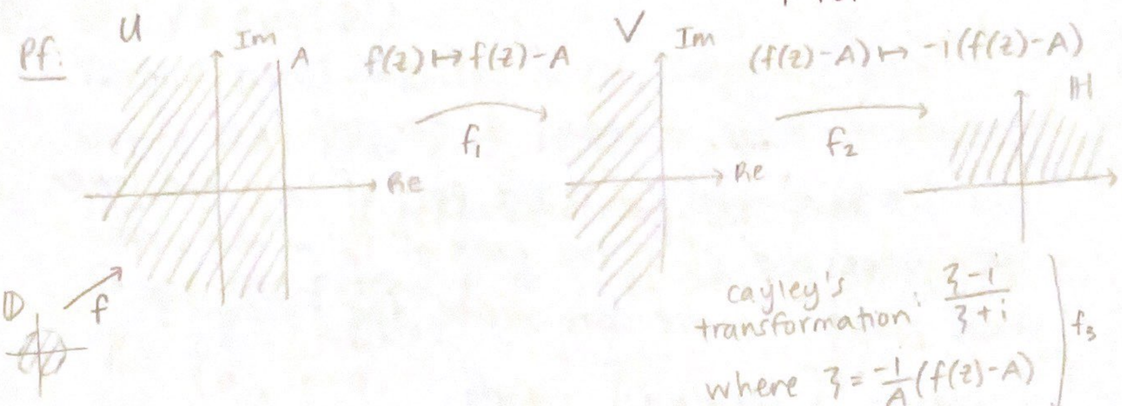
$$\Rightarrow \quad 2\pi i \quad = \quad 0 \quad + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Thus, we conclude that $\lim_{R\to+\infty} \int_{C_R} \frac{e^{iz}}{z}\,dz = 2\pi i$. □

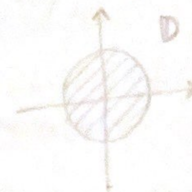
continued...

⑥ Let f be a holomorphic function on $\mathbb{D}$. Assume $f(0)=0$ and $\operatorname{Re} f \le A$ on $\mathbb{D}$ for some constant $A>0$. Show that $|f(z)| \le \dfrac{2A|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

<u>Pf.</u>



Observe that $-i(f(0)-A) = iA$, but we want $-i(f(0)-A) = i$, so write $\underbrace{\frac{-1}{A}(f(0)-A) = i}$



Why?

Say "Scaling by $\frac{1}{A}$, $f_4 : \mathbb{H} \to \mathbb{H}$, and composing $f_4 \circ f_3 \circ f_2 \circ f_1 \circ f$, we get"

$$\frac{\frac{-1}{A}(f(z)-A)-i}{\frac{-1}{A}(f(z)-A)+i} = \frac{-i(f(z)-A)-iA}{-i(f(z)-A)+iA}$$

$$= \frac{-if(z)+iA-iA}{-if(z)+iA+iA}$$

$$= \frac{-i(f(z))}{-i(f(z)-2A)}$$

$$= \frac{f(z)}{f(z)-2A}$$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ by $\varphi(z) = \dfrac{f(z)}{f(z)-2A}$

Then $\varphi(0) = \dfrac{f(0)}{f(0)-2A} = \dfrac{0}{0-2A} = 0$, and $|\varphi(z)| \le 1$.

By Schwarz's lemma, $|\varphi(z)| \le |z|$:

$$|\varphi(z)| = \left| \frac{f(z)}{f(z)-2A} \right| \le |z| \implies |f(z)| \le |z|\,|f(z)-2A|$$

$$\le |z| \cdot \left[|f(z)| + |2A| \right]$$

$$\le |z||f(z)| + |z| \cdot 2A$$

$$|f(z)| - |z||f(z)| \le |z| \cdot 2A$$

$$|f(z)|(1-|z|) \le 2A|z|$$

$$\Rightarrow |f(z)| \le \frac{2A|z|}{1-|z|} \quad \text{for all } z \in \mathbb{D}.$$

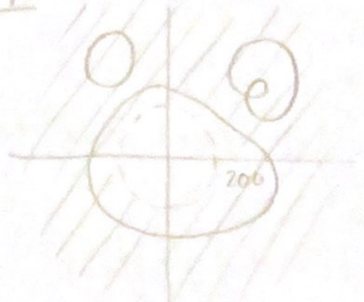
$\square$

...ued...

Is there a holomorphic function g defined on $\Omega = \{z \in \mathbb{C};\ |z| > 200\}$ such that $g'(z) = \dfrac{z^{51}}{\prod_{m=1}^{100}(z-m)}$ for all $z \in \Omega$? Prove your assertion.

It suffices to show that $\int_\gamma g'(z)\,dz = 0$ for all γ closed curves in Ω

Pf:



Case 1: γ is a closed curve in Ω that does not wind around 0

Then $\int_\gamma g'(z)\,dz = 0$ by Cauchy's theorem since g' is analytic in an open nbhd of γ that contains all points bounded by γ.

Case 2: γ winds around 0. WLOG, we may Assume $\gamma = re^{it}$, $0 \le t \le 2\pi$, $r > 200$.

By partial fraction decomposition, we have:

$$\frac{z^{51}}{\prod_{m=1}^{100}(z-m)} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \cdots + \frac{A_{100}}{z-100} \qquad \left(z^{99}\sum_{k=1}^{100} A_k + g(z)\right)$$

By equating the numerators, we have:

$z^{51} = A_1(z-2)(z-3)\cdots(z-100) + \ldots + A_{100}(z-1)(z-2)\cdots(z-99)$

Then $\sum A_k$ is the coeff. of z^{99} on the LHS so, $\sum A_k = 0$.

Note $\int_\gamma \frac{A_k}{z-k}\,dz = A_k(2\pi i)$ (since all poles contained in interior of γ)

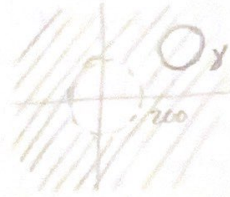
Thus, $\int_\gamma g'(z)\,dz = 2\pi i \cdot \sum_{i=1}^{100} A_k = 0$

$(z^{51} = A_1 z^{99} + \ldots + A_2 z^{99} + \ldots + A_{100} z^{99} + \ldots)$

Thus, we conclude that $g'(z)$ has a primitive in Ω. $\square$

OR

Pf: Recall that the function g' has a primitive if and only if $\int_\gamma g'(z)\,dz = 0$ for all closed contours in Ω.



For any closed contour whose interior is in Ω,
we have that $\int_\gamma g'(z)\,dz = 0$ by Cauchy's theorem.

Now if Ω^c is contained in the interior of a curve γ, consider the partial fraction decomposition:

$$\frac{z^{51}}{\prod_{m=1}^{100}(z-m)} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \dots + \frac{A_{100}}{z-100}.$$

Multiplying through by $\prod_{m=1}^{100}(z-m)$, we get

$$z^{51} = A_1(z-2)(z-3)\cdots(z-100) + \dots + A_{100}(z-1)(z-2)\cdots(z-99)$$

$$= z^{99}\sum_{k=1}^{100} A_1 + h(z),$$ where $h(z)$ is the remaining poly. of deg 98

Observe that z^{51} has no term of deg. 99, so the coeff of z^{99} on the RHS must be 0: $\sum_{k=1}^{100} A_k = 0$.

Now, since all poles are contained within γ, $\int_\gamma \frac{A_k}{z-k}\,dz = A_k(2\pi i)$.

Therefore, $\int_\gamma g'(z)\,dz = 2\pi i \sum_{k=1}^{100} A_k = 0$.

Thus, g' has a primitive, so there exists such function g.

$\square$