...uary 2021

) Let f be a nonconstant smooth function on $\mathbb{C}$ such that the set Γ given by $\Gamma = \{z \in \mathbb{C} : |f(z)| = 7\}$ is a smooth simple closed curve in $\mathbb{C}$. Denote by G the bounded region enclosed by Γ. Assume f is holomorphic in G. Prove that f has at least one zero in G. $\to$ f has no zeros in G

Pf. Assume f is nonzero in G, so $f(z) \neq 0 \ \forall z \in G$.

Since G is bounded and f is holomorphic in G and continuous on $\partial G = \Gamma$, by the maximum modulus principle we have that $|f|$ must attain a maximum on Γ.

Since we assumed that f is nonzero, by the minimum modulus principle, we have that $|f|$ must also attain a minimum on Γ.

But $|f(z)| = 7$ on Γ, so $\max\limits_{z \in \overline{G}} |f(z)| = \min\limits_{z \in \overline{G}} |f(z)| = 7$, which means that

$|f(z)| = 7 \ \forall z \in \overline{G}$.

The maximum modulus principle tells us that if $|f|$ attains a max. in G, then f is constant.

Since $|f(z)| = 7 = \max\limits_{z \in \overline{G}} |f(z)|$ in G (i.e., $|f|$ attains a maximum in G), we have that f is constant. $\lightning$

This is a contradiction because we assumed that f is nonconstant.

Therefore, f has at least one zero in G. $\square$

Continued.

② Let g be an entire function satisfying $\max_{\{|z| \le R\}} |g(z)| \le R^9$, for all $R \ge 200$.

Show that g is a polynomial of degree at most 9.

Pf. Since g is entire, we can write g as a convergent power series centered at $z=0$, so $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Fix $R \ge 200$.

By Cauchy's formula, $a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{g(z)}{z^{n+1}}\, dz$.

Thus, $|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{g(z)}{z^{n+1}}\, dz \right| \le \frac{1}{2\pi} \int_{|z|=R} \frac{|g(z)|}{|z|^{n+1}} |dz| \le \frac{1}{2\pi} \int_{|z|=R} \frac{|g(z)|}{R^{n+1}} |dz|$

$= \frac{1}{2\pi R^{n+1}} \int_{|z|=R} |g(z)||dz| \le \frac{1}{2\pi R^{n+1}} \int_{|z|=R} R^9 |dz| = \frac{R^9}{2\pi R^{n+1}} \int_{|z|=R} |dz|$

$\le \frac{R^9}{2\pi R^{n+1}} \cdot 2\pi R = \frac{R^9}{R^n} = \frac{1}{R^{n-9}} \to 0$ as $R \to \infty$ for $n > 9$.

↳ Since this holds for all $R \ge 200$, $(n>9)$ letting $R \to \infty$, we see that $\frac{1}{R^{n-9}} \to 0$.

Therefore, $a_n = 0$ for $n > 9$.

Thus, we conclude that g is a polynomial of degree at most 9.

□

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How many zeros counting multiplicities does the function $\psi(z) = z^8 - 6e^z + 5$ have in the region $\{z \in \mathbb{C} : |z| < 2\}$? Prove your assertion.

Pf: On $\partial\{z \in \mathbb{C} : |z| < 2\} \Rightarrow |z| = 2$, we have

$|z^8| = 2^8 = 256$

Let $z = re^{i\theta} = 2e^{i\theta} = 2\cos\theta + 2i\sin\theta$

$|6e^z| = |6e^{2\cos\theta + 2i\sin\theta}| = |6e^{2\cos\theta}e^{2i\sin\theta}| = |6e^{2\cos\theta}| = 6e^2$.

$|5| = 5$

Let $f(z) = z^8$ and $g(z) = -6e^z + 5$.

On $|z| = 2$, we have that $|g(z)| \le 6e^2 + 5 < 256 = |f(z)|$.

Therefore, by Rouché's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeros in $\{z \in \mathbb{C} : |z| < 2\}$.

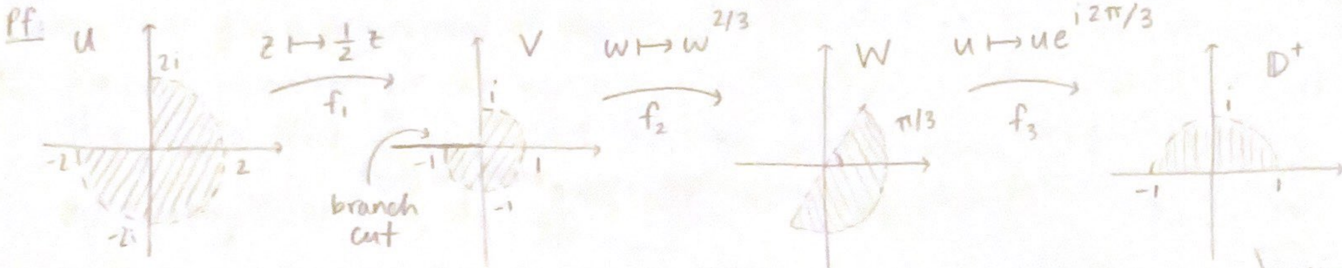
We have that $f(z) = z^8$ has a zero at $z = 0$ w/ multiplicity 8.

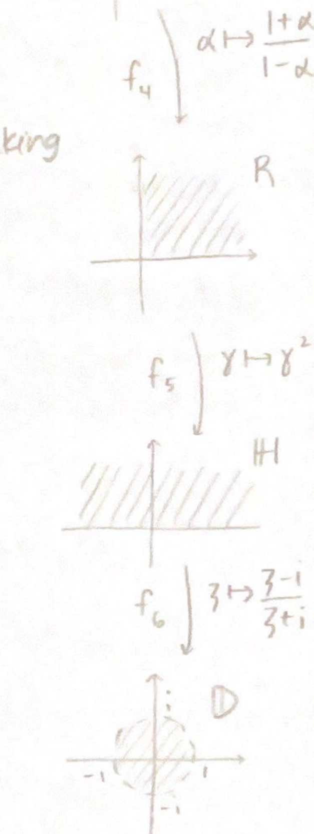
Therefore, $f(z) + g(z) = \psi(z) = z^8 - 6e^z + 5$ has 8 zeros counting multiplicities in $\{z \in \mathbb{C} : |z| < 2\}$. □

Continued...

④ Let $U = \{re^{i\theta} : 0 < r < 2, -\pi < \theta < \pi/2\}$. Explicitly describe a one-to-one conformal ma[p] from U onto the unit disk $\mathbb{D}$.

Pf:





Let $f_1 : U \to V$ by $f_1(z) = \frac{1}{2}z$

$f_2 : V \to W$ by $f_2(w) = w^{2/3}$. Then f_2 is analytic on V by taking the branch cut where $-\pi < \arg(z) < \pi$.

$f_3 : W \to \mathbb{D}^+$ by $f_3(u) = ue^{i\frac{2\pi}{3}}$

$f_4 : \mathbb{D}^+ \to R$ by $f_4(\alpha) = \frac{1+\alpha}{1-\alpha}$

$f_5 : R \to \mathbb{H}$ by $f_5(\gamma) = \gamma^2$

$f_6 : \mathbb{H} \to \mathbb{D}$ by $f_6(\zeta) = \frac{\zeta - i}{\zeta + i}$.

Let $f : U \to \mathbb{D}$ by $f(z) = (f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(z)$.

We have that f is one-to-one conformal since the composition of one-to-one conformal maps is one-to-one conformal. $\square$

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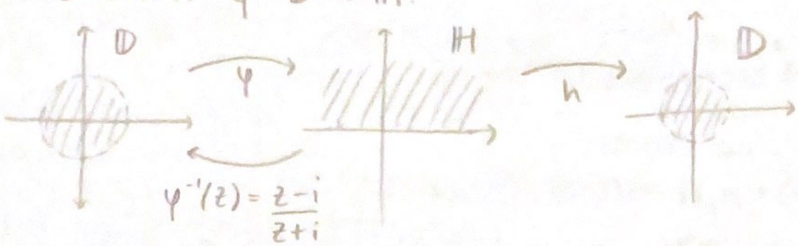
Let $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. For all holomorphic functions h in $\mathbb{H}$ such that $h(i) = 0$ and $|h(z)| < 1$ for all $z \in \mathbb{H}$, find the largest possible value of $|h(6i)|$.

Pf. Let $h : \mathbb{H} \to \mathbb{D}$, $h(i) = 0$, and $|h(z)| < 1$.

we want $\varphi : \mathbb{D} \to \mathbb{H}$.



Let $\frac{z-i}{z+i} = w \Rightarrow z - i = w(z+i) = wz + wi$

$z - wz = wi + i$

$z(1-w) = i(w+1)$

$z = \frac{i(w+1)}{1-w}$

So $\varphi(z) = \frac{i(z+1)}{1-z}$

Let $g : \mathbb{D} \to \mathbb{D}$ by $g(z) = (h \circ \varphi)(z)$.

Notice that $g(0) = h(\varphi(0)) = h(i) = 0$, and $|g(z)| < 1$.

Therefore, by Schwarz's lemma, we have that $|g(z)| \le |z| \ \forall z \in \mathbb{D}$.

So we have, $|g(z)| \le |z|$

$\Rightarrow \left| h\left(\frac{i(z+1)}{1-z}\right) \right| \le |z| \Rightarrow |h(z)| \le \left| \frac{z-i}{z+i} \right| \Rightarrow |h(6i)| \le \left| \frac{6i-i}{6i+i} \right| = \left| \frac{5i}{7i} \right| = \frac{5}{7}.$

$\left[\begin{array}{l} \hookrightarrow \text{let } \zeta = \frac{i(z+1)}{1-z} \\ \Rightarrow z = \frac{\zeta - i}{\zeta + i} \end{array} \right]$

Therefore, the largest possible value of $|h(6i)|$ is $\frac{5}{7}$.

There exists $f(z) = \frac{z-i}{z+i}$ $(f : \mathbb{H} \to \mathbb{D})$ s.t. $|f(6i)| = \frac{5}{7}$

so this bound is sharp. (Sharp means the upper bound is attained).

$\square$

continued...

⑥ Let $C = \{z \in \mathbb{C} : |z| = 10^5\}$ with the positive direction. Evaluate the integral

$$\frac{1}{2\pi i}\oint_C \frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)}\,dz.$$

Let $f(z) = \frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)}$. Observe that f has simple poles at $z=k$ for $1 \le k \le 2021$.

Pf: Write $\frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \dots + \frac{A_{2021}}{z-2021}$ (partial fraction decomposition)

Then $z^{2020} = A_1(z-2)\cdots(z-2021) + A_2(z-1)(z-3)\cdots(z-2021) + \dots + A_{2021}(z-1)\cdots(z-2020)$

Notice that the coefficient of z^{2020} is 1 on the LHS and is $\sum_{k=1}^{2021} A_k$ on the RHS, So we have that $\sum_{k=1}^{2021} A_k = 1$.

We also have that $A_k = \lim_{z\to k} \frac{(z-k)\,z^{2020}}{\prod_{k=1}^{2021}(z-k)} = \frac{k^{2020}}{\prod_{\substack{j=1 \\ j\neq k}}^{2021}(k-j)}$.

By the partial fraction decomposition,

$$\frac{1}{2\pi i}\int_C \frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)}\,dz = \frac{1}{2\pi i}\left[\int_C \frac{A_1}{z-1}\,dz + \dots + \int_C \frac{A_{2021}}{z-2021}\,dz\right]$$

Note that $\int_C \frac{A_j}{z-j}\,dz = A_j \int_C \frac{1}{z-j}\,dz$, where $c = re^{it}$, $0 \le t \le 2\pi$, $r = 10^5$.

Observe that all poles are bounded by C.

So by the residue theorem, $A_j \int_C \frac{1}{z-j}\,dz = 2\pi i \cdot A_j$.

Therefore, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_C f(z)\,dz &= \frac{1}{2\pi i}\left[\sum_{j=1}^{2021}\int_C \frac{A_j}{z-j}\,dz\right] \\
&= \frac{1}{2\pi i}\left[2\pi i \sum_{j=1}^{2021} A_j\right] \\
&= \frac{1}{2\pi i}\left[2\pi i \cdot 1\right] \\
&= 1.
\end{aligned}$$

Thus, we conclude that $\frac{1}{2\pi i}\oint_C \frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)}\,dz = 1.$ □