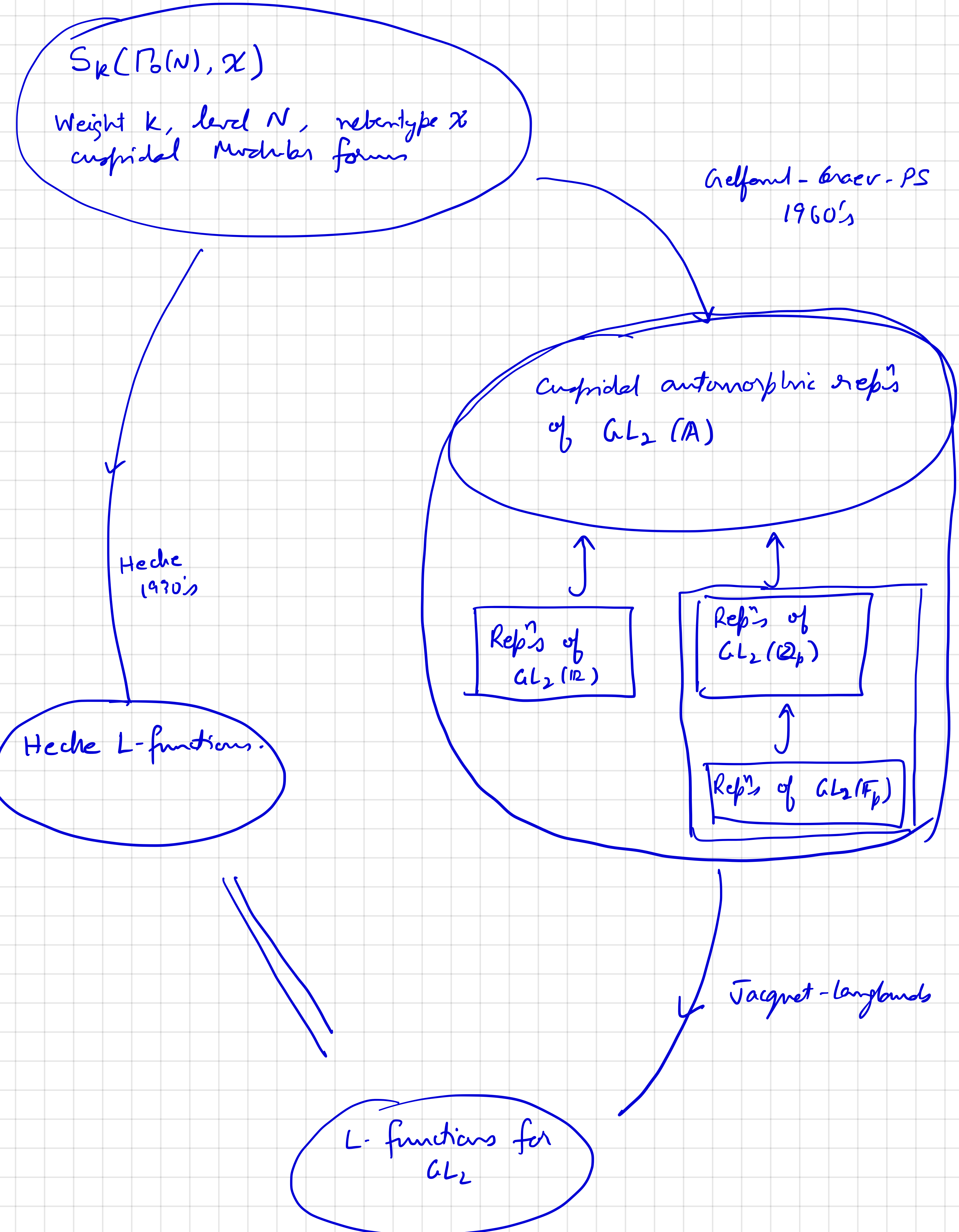
Basic Lie Theory.

31/Aug/26

1. $G = K \cdot A \cdot N$

2. $Z(\mathcal{U}(\mathfrak{g}))$

§0 The aim of this Seminar:



§1

$G = GL_n(\mathbb{R})$ = $n\times n$-invertible matrices$/\mathbb{R}$.

$K = O(n) = \{ g \in GL_n(\mathbb{R}) \mid {}^t g \cdot g = I_n \}$

$$A = \left\{ \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} : a_i \in \mathbb{R}_+^\times \right\}$$

$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$
$\mathbb{R}_+^\times = \mathbb{R}_{>0}$

$$N = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \text{ in } GL_n(\mathbb{R}) \right\}$$

Theorem:

$$\boxed{\text{The map } (k, a, n) \longmapsto kan \text{ gives a diffeomorphism: } K \times A \times N \xrightarrow{\sim} G.}$$

"Pf":

This is Gram-Schmidt orthogonalization.

$K = \{ g \in G \mid \langle gv, gw \rangle = \langle v, w \rangle \}$ $\qquad \langle v, w \rangle = \sum_{i=1}^n v_i w_i$

Corollaries / Exercises:

① The inclusion $K \hookrightarrow G$ induces an isomorphisms:

$$\pi_0(K) \cong \pi_0(G), \qquad \pi_1(K) \cong \pi_1(G)$$

② $\pi_0(K) \cong \mathbb{Z}/2\mathbb{Z}$ $\qquad$ (Ex: $SO(n)$ is connected $\quad SO(n)/SO(n-1) = S^{n-1}$)

$\therefore \pi_0(GL_n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$

③ $\pi_1(SL_2(\mathbb{R})) = \mathbb{Z}$

④ $$SO(2) \times \underbrace{\left\{\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right\} \times \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}}_{\mathbb{H}} \cong SL_2(\mathbb{R})$$

$\mathbb{H}$ - upper half plane.

§2 $Z(\mathcal{U}(\mathfrak{g}))$

$G = SL_2(\mathbb{R}),\quad GL_2(\mathbb{R}),\quad GL_n(\mathbb{R}), \ldots$

$\mathfrak{g} = \mathrm{Lie}(G)$ = Lie algebra of G $/_{\mathbb{R}}$.

(Sometimes : $\mathfrak{g}_0 = \mathrm{Lie}(G)$, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$.)

$G = SL_2(\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}\cdot\langle X, Y, H\rangle$

$\left\{ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$ – Standard $\mathfrak{sl}_2$ triple

We are interested in representations of G

Representation of G
$G \xrightarrow{\pi} GL(V)$

$\xrightarrow{\text{Differentiate}}$

Representation of the Lie algebra $\mathfrak{g}$
$\mathfrak{g} \longrightarrow \mathrm{End}(V)$

Defn: A Lie algebra rep^n. $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a linear map

s.t. $\pi([X, Y]) = [\pi(X), \pi(Y)]$

$\mathfrak{g}/_{\mathbb{C}}$ is "non-associate" 'non-commutative'.

$\mathcal{U}(\mathfrak{g})/_{\mathbb{C}}$ = associative algebra such that

$\{\text{Rep}^n. \text{ of } \mathfrak{g}\} \xleftrightarrow{\sim} \{\text{Modules of } \mathcal{U}(\mathfrak{g})\}$.

Defn: Let $\mathfrak{g}/\mathbb{C}$ be a Lie algebra.

A universal enveloping algebra of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is an associate algebra with a linear map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that (i) $i([x,y]) = i(x)\,i(y) - i(y)\,i(x)$

(ii) For any associative algebra $\mathcal{U}$ and a linear map $j : \mathfrak{g} \to \mathcal{U}$ satisfying (i), there is a unique homomorphism $\phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}$ s.t.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dotted, "\phi"] \\ & \mathcal{U} \end{tikzcd}$$

__Construction:__ $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) \big/ \langle \{ [x,y] - x \otimes y + y \otimes x : x, y \in \mathfrak{g} \} \rangle$

(P-B-W Theorem. (Poincare, Birkhoff, Witt))

$\mathfrak{g}$ has basis $\{x_1, \dots, x_n\}$ then $\mathcal{U}(\mathfrak{g})$ has basis $\{ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : a_j \in \mathbb{Z}_{\geq 0} \}$.

__Cor__: $\dim_{\mathbb{C}} (\mathcal{U}(\mathfrak{g}))$ is countable.

Even though a semisimple Lie algebra $\mathfrak{g}$ has no center, the enveloping algebra $\mathcal{U}(\mathfrak{g})$ has nontrivial center.

__Harish-Chandra's Theorem__:

$$\boxed{Z(\mathcal{U}(\mathfrak{g})) \cong \mathrm{Sym}(\mathfrak{h})^W}$$

$\mathfrak{g}$ - reductive Lie alg., $\mathfrak{sl}_2$

$\mathfrak{h}$ = Cartan subalg. $\left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$

eg: $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} = \mathbb{C} \cdot \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, $\mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[x]$, polynomial alg. in one variable

$W = \mathbb{Z}/2$, $\mathrm{Sym}(\mathfrak{h})^W = \mathbb{C}[\Delta]$ = polynomial alg. in one-variable

Δ - Casimir elt.

§ Why care about the center?

Schur's Lemma:

Suppose (π, V) is an irreducible rep^n. of a finite group G.

(Then V has to be finite dimensional.) The space of intertwining operators

$$Hom_G(V,V) = \{ \varphi : V \longrightarrow V \mid \varphi(\pi(g)v) = \pi(g)\varphi(v) \}$$

is one dimensional.

Pf: $T \in Hom_G(V,V)$, T has an eigenvalue, say $\lambda \in \mathbb{C}$.

Then $Ker(T - \lambda I_V)$ is a nontrivial G-stable subspace

$\therefore$ $Ker(T - \lambda 1_V) = V$ or $T = \lambda \cdot 1_V$

Dixmier's Lemma:

Suppose S is a family of operators acting irreducibly on a countable dimensional vector space V. Suppose $T \in End(V)$ commutes with each $S \in S$, i.e., $TS = ST$.

Then T is a scalar

Pf: (Jacquet's Proof) $\forall \lambda \in \mathbb{C}$, $(T - \lambda 1)$ is injective.

V is a (faithful) module for $\mathbb{C}(x)$ - rational functions$/\mathbb{C}$ in one variable.

V is countable dimension.

$\mathbb{C}(x)$ is uncountable dimension.

$\}$ $\otimes$. Why?

Example/Exercise:

$\mathfrak{sl}_2/\mathbb{C} = \mathbb{C}\langle H, X, Y\rangle$

$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[H, X] = 2X$

$[H, Y] = -2Y$

$[X, Y] = H$

$\Big\}$ Standard $\mathfrak{sl}_2$ triple.

(i) $\Delta = XY + YX + \frac{1}{2}H^2$ $\qquad Z(\mathcal{U}(\mathfrak{g})) = \mathbb{C}[\Delta]$.

(Casimir)

(ii) Let $V = \mathbb{C}^2$ $\quad \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathrm{End}(V)$ the standard rep^n.

Let $n \geq 1$, Consider $\mathrm{Sym}^n(V)$

$\mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathrm{End}(\mathrm{Sym}^n(V))$ — Fact: It is the unique irreducible rep^n of $\dim(n+1)$.

compute the action of Δ on $\mathrm{Sym}^n(V)$.

(We will need this computation later!)

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