Upper Half plane.

31/Aug/26

- $\mathbb{H} = SL_2(\mathbb{R})/SO(2)$
- Cayley Transform: $f : \mathbb{H} \longrightarrow \mathbb{K}$
- Elliptic, Parabolic, Hyperbolic elements
- $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ $\quad$ $dv(z) = \dfrac{dx\,dy}{y^2}$

§1 Upper half plane / Unit Disk:

§§ Linear Fractional Transformations.

- $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ = Riemann Sphere
- $GL_2(\mathbb{C}) \circlearrowright \mathbb{P}$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \in \mathbb{C} \cup \{\infty\}, \quad \alpha z = \frac{az+b}{cz+d}.$$

- $$\boxed{j(\alpha, z) = cz + d}$$

- $z \longmapsto \alpha z$ is an action: $\boxed{\alpha(\beta z) = \alpha\beta(z)}$

$j(\alpha, z)$ has a cocycle condition: $\boxed{j(\alpha\beta, z) = j(\alpha, \beta z)\, j(\beta, z)}$

Pf
$$\alpha \cdot \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = j(\alpha, z) \begin{bmatrix} \alpha z \\ 1 \end{bmatrix}$$

$$\alpha\left(\beta \begin{bmatrix} z \\ 1 \end{bmatrix}\right) = \alpha\left(j(\beta, z) \begin{bmatrix} \beta z \\ 1 \end{bmatrix}\right) = j(\beta, z)\, j(\alpha, \beta z) \cdot \begin{bmatrix} \alpha(\beta(z)) \\ 1 \end{bmatrix}$$

$$\parallel$$

$$\alpha\beta \begin{bmatrix} z \\ 1 \end{bmatrix} = j(\alpha\beta, z) \cdot \begin{bmatrix} \alpha\beta(z) \\ 1 \end{bmatrix}$$

Lemma:

$$\boxed{\alpha \in GL_2(\mathbb{C}) \text{ maps circles/lines on } \mathbb{C} \text{ to circles/lines on } \mathbb{C}.}$$

Pf
$$\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C}),$$

$$C_\beta = \{ z \in \mathbb{P} \mid |\beta(z)| = 1 \}$$

$$z \in C_\beta \cap \mathbb{C} \iff |az+b| = |cz+d| \begin{cases} (a=c) & \text{Line} \\ (a \neq c) & \text{Circle} \end{cases}$$

$$\alpha(C_\beta) = C_{\beta\alpha^{-1}}$$

$$z \in C_{\beta\alpha^{-1}} \iff |\beta\alpha^{-1}(z)| = 1 \iff \alpha^{-1}(z) \in C_\beta \iff z \in \alpha(C_\beta).$$

§§ $\mathbb{H}$ & $\mathbb{K}$:

- $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

  $\mathbb{K} = \{ z \in \mathbb{C} \mid |z| < 1 \}$

Lemma (Cayley Transform)

$$\boxed{\rho : \mathbb{H} \longrightarrow \mathbb{K} \qquad \rho(z) = \frac{z-i}{z+i} \quad \text{is a complex analytic isomorphism.}}$$

Pf: $z \in \mathbb{H} \iff z$ is closer to i than to $-i$

$\iff |z-i| < |z+i| \iff |\rho(z)| < 1$

- $\left.\begin{array}{l} \alpha \in GL_2(\mathbb{R})^+ = \{ g \in GL_2(\mathbb{R}) \mid \det(g) > 0 \} \\ z \in \mathbb{H} \end{array}\right\} \Rightarrow \alpha(z) \in \mathbb{H}.$

(check: $\operatorname{Im}(\alpha(z)) = \dfrac{\det(\alpha)}{|j(\alpha, z)|^2} \operatorname{Im}(z).$)

Theorem:

$$\boxed{GL_2(\mathbb{R})^+/\mathbb{R}^\times \cong SL_2(\mathbb{R})/\{\pm I\} \cong \operatorname{Aut}(\mathbb{H})}$$

"Pf": • $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. $\quad \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \cdot i = iy$

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \cdot i = x + iy$

• $\operatorname{Stab}_{SL_2(\mathbb{R})}(i) = SO(2)$

• $\phi \in \operatorname{Aut}(\mathbb{H})$. Assume that $\phi(i) = i$.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\phi} & \mathbb{H} \\ \rho \downarrow & & \downarrow \rho \\ \mathbb{K} & \xrightarrow{\psi} & \mathbb{K} \end{array}$$

$\psi = \rho \phi \rho^{-1} : \mathbb{K} \to \mathbb{K}$, $\psi(0) = 0$,) Schwarz's Thm.

Then ψ is a rotation.

If $\gamma(z) = e^{i\theta} z$, $\gamma = \rho^{-1}\phi\rho \Rightarrow \phi = \rho\gamma\rho^{-1} \Rightarrow \phi = \begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{bmatrix}$

Theorem:

> Suppose G is a locally compact, T_2, second countable, topological group, acting continuously and transitively on a locally compact T_2 space X. Then
>
> $$\begin{aligned} G/\mathrm{Stab}_G(x) &\xrightarrow{\sim} X \quad \text{is a homeomorphism.} \\ g &\longmapsto g\cdot x \end{aligned}$$

Cor:

$$\boxed{SL_2(\mathbb{R})/SO(2) \xrightarrow{\sim} \mathbb{H}. \quad \text{homeomorphism}}$$

§ Elliptic / Parabolic / Hyperbolic Elements:

Defn: $\alpha \in GL_2(\mathbb{R})^+$. Assume α is not a scalar. We say:

α is elliptic $\iff$ $tr(\alpha)^2 < 4\det(\alpha)$

α is parabolic $\iff$ $tr(\alpha)^2 = 4\det(\alpha)$

α is hyperbolic $\iff$ $tr(\alpha)^2 > 4\det(\alpha)$

- Char. poly. of α: $T^2 - tr(\alpha)\cdot T + \det(\alpha)$

  Discriminant: $tr(\alpha)^2 - 4\det(\alpha)$ $\quad (a+d)^2 - 4(ad-bc)$
  $(a-d)^2 + 4bc$

- Fixed points of α: $\alpha(z) = z \iff \dfrac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0$

  Discriminant: $(d-a)^2 + 4bc$

Theorem:

Suppose $\alpha \in GL_2(\mathbb{R})^+$ is not a scalar

(i) α is elliptic $\iff$ α has two fixed point, $z_0, \bar{z}_0$ with $z_0 \in \mathbb{H}$

(ii) α is parabolic $\iff$ α has a unique fixed point in $\mathbb{R} \cup \{\infty\}$

(iii) α is hyperbolic $\iff$ α has two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

Notation: $x, x' \in \mathbb{R} \cup \{\infty\}$

$$GL_2^+(\mathbb{R})_x^{(p)} = \{\alpha \in GL_2^+(\mathbb{R}) : \alpha \text{ scalar or } \alpha(x) = x \ \& \ \alpha \text{ parabolic}\}$$

$$GL_2^+(\mathbb{R})_{x,x'} = \{\alpha \in GL_2^+(\mathbb{R}) : \alpha(x) = x, \ \alpha(x') = x'\}$$

Examples:

$$GL_2^+(\mathbb{R})_\infty^{(p)} = \left\{ \begin{bmatrix} a & \\ & a \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R}^\times, \ b \in \mathbb{R} \right\}$$

$$GL_2^+(\mathbb{R})_{0,\infty} = \left\{ \begin{bmatrix} a & \\ & d \end{bmatrix} : ad > 0 \right\}$$

$$GL_2^+(\mathbb{R})_i = \left\{ \begin{bmatrix} a & \\ & a \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \right\}.$$

§ Invariant metric / Measure on $\mathbb{H}$.

Defn (i) f is a differentiable fn. on $\mathbb{H}$, then

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

(ii) $\alpha \in GL_2(\mathbb{R})^+$, $f \in C^1(\mathbb{H})$,

$$(f \circ \alpha)(z) = f(\alpha(z))$$

$$(df) \cdot \alpha = d(f \circ \alpha)$$

Eg. $dz = dx + i\,dy$ $\qquad z = x + iy$, $\dfrac{\partial z}{\partial x} = 1$, $\dfrac{\partial z}{\partial y} = i$

$$d\bar{z} = dx - i\,dy$$

$$d(z \circ \alpha) = \frac{d(\alpha(z))}{dz}\, dz$$

Defn Metric on $\mathbb{H}$ $\qquad \boxed{ds^2 = \dfrac{dx^2 + dy^2}{y^2}}$

e.g. $\varphi : [0,1] \longrightarrow \mathbb{H}$ is a smooth curve. $\varphi(t) = x(t) + i\,y(t)$.

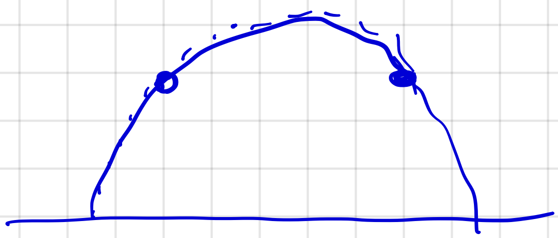
$$l(\varphi) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

Ex: compute the length of $i\mathbb{R} \hookrightarrow \mathbb{H}$

Defn Measure on $\mathbb{H}$: $\qquad \boxed{d\nu(z) = \dfrac{dx\,dy}{y^2}}$

Facts:

① Any two points on $\mathbb{H}$ are joined by a unique geodesic which is part of a circle orthogonal to the real axis or a line orthogonal to the real axis



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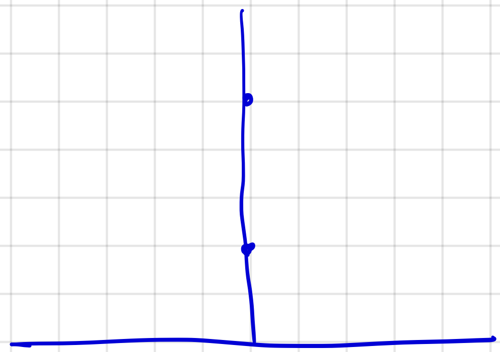
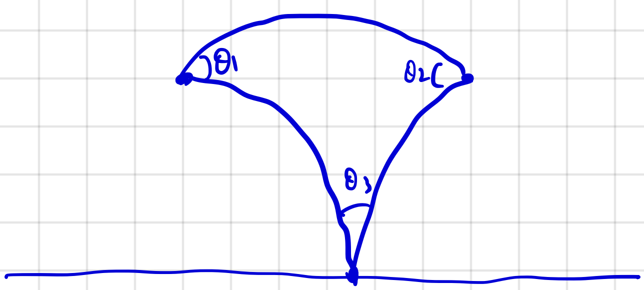


② Let D be the interior of a triangle in $\mathbb{H}$ with vertices on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ with interior angles $\theta_1, \theta_2, \theta_3$. Then $\text{area}(D) = \pi - (\theta_1 + \theta_2 + \theta_3)$



③ Measure on $\mathbb{H}$ as a homogeneous space: $SL_2(\mathbb{R}) \longrightarrow SL_2(\mathbb{R})/SO(2) \cong \mathbb{H}$.

If $f: \mathbb{H} \longrightarrow \mathbb{C}$ is a measurable fn. then so is $\alpha \longmapsto f(\alpha \cdot i)$ on $SL_2(\mathbb{R})$

$$\int_{\mathbb{H}} f(z) \cdot dv(z) = \int_{SL_2(\mathbb{R})} f(\alpha(i))\, d\alpha$$

$d\alpha$ = Haar measure on $SL_2(\mathbb{R})$