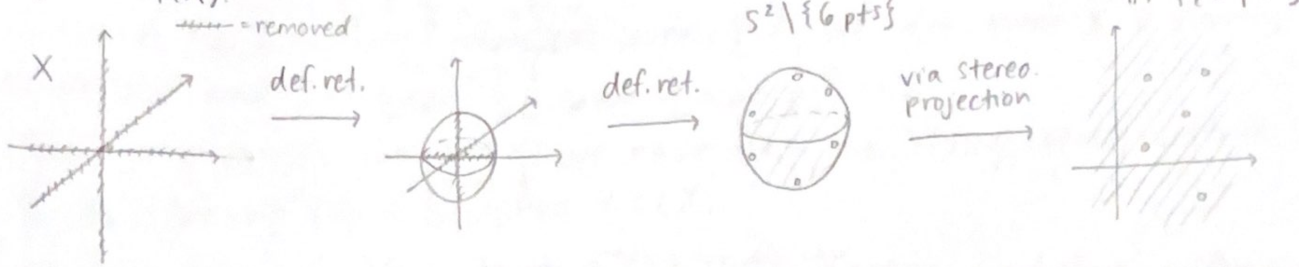
...st 2013

① Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Compute $\pi_1(X)$.

Pf:



Using induction, we will show that $\pi_1(\mathbb{R}^2\setminus\{k \text{ pts}\}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k\text{-copies}}$

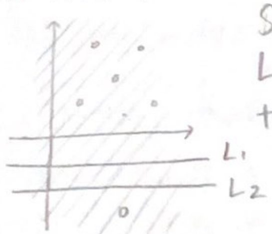
Base case: $k=1$: $\mathbb{R}^2\setminus\{1 \text{ pt}\}$

$\pi_1(\mathbb{R}^2\setminus\{1\text{ pt}\}) = \pi_1(S^1) = \mathbb{Z}$.

Induction hypothesis: Suppose that $\pi_1(\mathbb{R}^2\setminus\{k-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k-1 \text{ copies}}$.

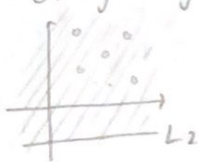
We WTS $\pi_1(\mathbb{R}^2\setminus\{k \text{ pts}\}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k\text{-copies}}$

$\mathbb{R}^2\setminus\{k \text{ pts}\}$



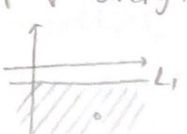
Since the k pts are distinct, there is space between them. Let L_1 and L_2 be two parallel lines between 1 point and the remaining $k-1$ pts.

Let U = everything above L_2 ($k-1$ pts removed)



U is open and path-connected.
By our induction hypothesis
$\pi_1(U) = \pi_1(\mathbb{R}^2\setminus\{k-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k-1 \text{ copies}}$

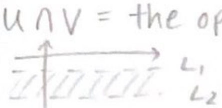
Let V = everything below L_1 (1 point removed)



V is open and path-connected.
By our base case, $\pi_1(V) = \pi_1(\mathbb{R}^2\setminus\{1 \text{ pt}\}) = \mathbb{Z}$

Observe that $U \cup V = X$.

$U \cap V$ = the open strip between L_1 and L_2



This is convex, so $\pi_1(U\cap V) = 0$.

continued...

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(\mathbb{R}^2 \setminus \{k \text{ pts}\}) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V)$

$$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-1 \text{ copies}} * \mathbb{Z}$$

$$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k\text{-copies}}$$

Therefore, $\pi_1(X) = \pi_1(\mathbb{R}^2 \setminus \{5 \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{5 \text{ copies}}$.

$\square$

...nued...

(?) Suppose a space X has a compact universal covering $\tilde{X}$. Show that the fundamental group of X is finite.

Pf: Since X has a compact universal covering $\tilde{X}$, we have that $\tilde{X}$ is simply connected and there exists a covering map $p: \tilde{X} \to X$.

Since $\tilde{X}$ is simply connected, we have that the lifting correspondence $\Phi: \pi_1(X, x) \to p^{-1}(x)$ is bijective $\forall x \in X$.

So it suffices to show that $p^{-1}(x)$ is finite.

We want an open cover of X.

Since p is a covering map, for each $x \in X$, $\exists U_x$ an evenly covered nbhd.

So $X = \bigcup_{x \in X} U_x = \bigcup_{i=1}^{n} U_{x_i}$ since X is compact ($p: \tilde{X} \to X$ is cts & surj. $\Rightarrow p(\underset{\text{cpt}}{\tilde{X}}) = X$, so $X = \text{cpt}$)

By taking p^{-1} of both sides we get $\underset{\tilde{X}}{p^{-1}(X)} = \bigcup_{i=1}^{n} p^{-1}(U_{x_i})$.

Since U_{x_i} are evenly covered nbhds, we have that $p^{-1}(U_{x_i}) = \bigcup_{\alpha \in J_i} V_i^\alpha$, where the V_i^α are an open collection of pairwise disjoint sets and satisfy $p: V_i^\alpha \to U_{x_i}$ homeomorphism.

We can write $p^{-1}(X)$ as $p^{-1}(X) = \tilde{X} = \bigcup_{i=1}^{n} p^{-1}(U_{x_i}) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in J_i} V_i^\alpha$.

Observe that $|p^{-1}(x) \cap V_i^\alpha| \leq 1$ (b/c $p|_{V_i^\alpha}$ is a homeo. (inj.))

There exists a finite collection of V_i^α's such that $\tilde{X} = \bigcup_{j=1}^{k} V_j$, where $p|_{V_j}$ is injective for all j.

$$p^{-1}(x) \cap \tilde{X} = \bigcup_{j=1}^{k} V_j \cap p^{-1}(x) \Rightarrow p^{-1}(x) = \underbrace{\bigcup_{j=1}^{k} V_j \cap p^{-1}(x)}_{\text{This can have at most } k \text{ points because } |p^{-1}(x) \cap V_i^\alpha| \leq 1.}$$

Therefore, $|p^{-1}(x)| \leq k \Rightarrow p^{-1}(x)$ is finite.

Thus, since Φ is a bijection, we have that $\pi_1(X)$ is finite.

$\square$

continued...

(3) Consider the following equivalency relation on $\mathbb{R}$: $x \sim y$ if $x-y$ is rational. Let X be the quotient space $\mathbb{R}/\sim$ with the quotient topology. Describe the topology X as explicitly as you can, and decide if X is Hausdorff. Justify your answer.

Pf: Let $X = \mathbb{R}/\sim$, and let $q: \mathbb{R} \to X$ be the quotient map.

Consider $[0]$ and $[\pi]$. We WTS that every nbhd of $[0]$ intersects every nbhd of $[\pi]$ (i.e., that X is not Hausdorff).

Let U be any open nbhd of $[0]$:

Since q is a quotient map, we know that

U is open in $q(\mathbb{R}) = X$ iff $q^{-1}(U)$ is open in $\mathbb{R}$.

So we know that $q^{-1}(U) \subseteq \mathbb{R}$ is open and contains 0.

If $x \in q^{-1}(U)$ and $x \sim y$, then $y \in q^{-1}(U)$ ($q^{-1}(U)$ is saturated)

Therefore, $q^{-1}(U)$ is closed under the equivalence relation $\sim$.

Since $q^{-1}(U) \subseteq \mathbb{R}$ is open and $0 \in q^{-1}(U)$, let $(-a,a) \subseteq q^{-1}(U)$ for some $a > 0$.

If $y \in \mathbb{R}$, we WTS $y \in q^{-1}(U)$.

$\mathbb{Q}$ is dense in $\mathbb{R}$, so let $z \in \mathbb{Q}$ be such that $|y-z| < a$.

Then $-a < y-z < a \Rightarrow y-z \in q^{-1}(U)$.

So $y \sim (y-z)$ because z is rational, so $y \in q^{-1}(U)$.

Since y was arbitrary, we have that $\mathbb{R} = q^{-1}(U)$.

Taking q of both sides, we get $\underbrace{q(\mathbb{R})}_{X} = q(q^{-1}(U)) = U$ b/c q is surj.

Therefore, $X = U$. U is an arbitrary nbhd of 0.

$[\pi] \in X = U$, so U intersects every nbhd of π.

Thus, we conclude that X is not Hausdorff.

- We can repeat this argument with some α in place of 0 to see that the only nbhd of $[\alpha]$ is X.

This implies that X is the only nonempty open set, i.e., that X has the trivial topology (only open sets are $\emptyset$, X).

$\square$

ued...

Let $\mathbb{D}$ be the closed unit disk in the plane and $f: \mathbb{D} \to \mathbb{D}$ is a homeomorphism. Show that f maps the unit circle $S^1 \subset \mathbb{D}$ into itself.

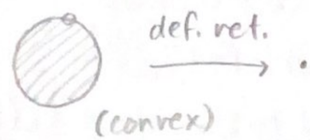
Pf: We will argue by contradiction.

Assume $z \in S^1$ s.t. $f(z) \notin S^1$.

The restriction $f: \mathbb{D}\setminus\{z\} \to \mathbb{D}\setminus\{f(z)\}$ is still a homeomorphism.

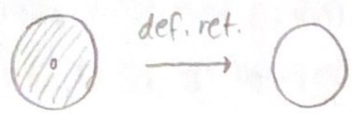
Observe that

$\mathbb{D}\setminus\{z\}$



$\pi_1(\mathbb{D}\setminus\{z\}) = 0$

$\mathbb{D}\setminus\{f(z)\}$



$\pi_1(\mathbb{D}\setminus\{f(z)\}) = \mathbb{Z}$

So restricted f is a homeomorphism between two spaces that have different fundamental groups. $\lightning$

This is a contradiction, so we conclude that $z \in S^1$ implies $f(z) \in S^1$. $\square$

continued...

(5) (a) Prove that if a path-connected, locally path-connected space X has finite fundamental group, then any map $f: X \to S^1$ is homotopic to a constant map (null-homotopic).

Pf: We would like to use the general lifting lemma.

Observe that X is path-connected and locally path-connected.

Let $p: \mathbb{R} \to S^1$ (the exp. map) be a covering map.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

First we want to show that there exists a lift $\tilde{f}: X \to \mathbb{R}$.

It remains to show that $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$.

Observe that $\pi_1(\mathbb{R}) = 0$, so we need to show that $f_*(\pi_1(X)) = 0$.

Since $\pi_1(X)$ is finite, so is $f_*(\pi_1(X))$, which is contained in $\pi_1(S^1) = \mathbb{Z}$. The only finite subgp of $\mathbb{Z}$ is 0, so $f_*(\pi_1(X)) = 0 \subseteq p_*(\pi_1(\mathbb{R}))$ ✓

Therefore, $\exists$ a lift $\tilde{f}: X \to \mathbb{R}$.

Recall that any continuous map into a contractible space is null-homotopic.
Since $\tilde{f}$ is continuous and $\mathbb{R}$ is contractible, we have that $\tilde{f}$ is null-homotopic.
Let H be a homotopy between $\tilde{f}$ and a constant map, then $p \circ H$ is a homotopy between f and a constant.

Therefore, if $\tilde{f}$ is null-homotopic, then so is f.

Since $\tilde{f}$ is null-homotopic, so is f.

Thus, any map $f: X \to S^1$ is null-homotopic.

□

nued...

) Prove that any map $f: \mathbb{RP}^2 \to \mathbb{T}^2$, from the real projective plane to the torus, is homotopic to a constant map (null-homotopic).

Pf: We want to use the general lifting lemma to show that $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}^2$ exists.

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{T}^2 \end{tikzcd}$$

Observe that $\mathbb{RP}^2$ is path-conn. (b/c it is the cts image of S^2 which is path-conn., and the cts image of path-conn. is path-conn.).

Observe that $\mathbb{RP}^2$ is locally path-conn. (b/c $q: S^2 \to \mathbb{RP}^2$ is a local homeomorphism).

Recall that if $f_1: E_1 \to X_1$ and $f_2: E_2 \to X_2$ are covering maps, then $f_1 \times f_2: E_1 \times E_2 \to X_1 \times X_2$ given by $(f_1 \times f_2)(e_1, e_2) = (f_1(e_1), f_2(e_2))$ is a covering map.

Since $\mathbb{T}^2 = S^1 \times S^1$, we have that $p: \mathbb{R}^2 \to \mathbb{T}^2$ $(\mathbb{R} \times \mathbb{R} \to S^1 \times S^1)$ is a covering map.

It remains to show that $f_*(\pi_1(\mathbb{RP}^2)) \subseteq p_*(\pi_1(\mathbb{R}^2))$.

Observe that $\pi_1(\mathbb{R}^2) = 0$, so we WTS $f_*(\pi_1(\mathbb{RP}^2)) = 0$.

Since $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ (is finite), we know that $f_*(\pi_1(\mathbb{RP}^2))$ is finite.

We know that $f_*(\pi_1(\mathbb{RP}^2)) \subseteq \pi_1(\mathbb{T}^2) = \mathbb{Z}^2$. The only finite subgroup of $\mathbb{Z}^2$ is 0, so $f_*(\pi_1(\mathbb{RP}^2)) = 0 \subseteq p_*(\pi_1(\mathbb{R}^2))$ ✓

Therefore, by the general lifting lemma the lift $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}$ exists.

Since $\mathbb{R}^2$ is contractible and any cts map into a contractible space is null-homotopic, we have that $\tilde{f}$ is null-homotopic.

Since $\tilde{f}$ is null-homotopic, we have that f is null-homotopic.

□