August 2015

① Let X be a topological space and let A be a subset of X. Either prove the following statement, or give a counter-example.

1. If A is connected, then the closure $\bar{A}$ is connected.

Pf: Let $\bar{A} \subseteq U \cup V$ where U, V are open, nonempty, and disjoint. ($\bar{A} \cap U$, $\bar{A} \cap V$ nonempty).

Since A is connected, $A \subseteq U$ or $A \subseteq V$.

WLOG, suppose $A \subseteq U$. We WTS $\bar{A} \cap V = \emptyset$ ($x \in \bar{A} \Rightarrow x \in U$).

If $x \in A$, then $x \in U$.

If x is a limit point of A, then every nbhd of x intersects $A \setminus \{x\}$.

So V is an open nbhd s.t. $U \cap V = \emptyset$ and $x \in V$, so $V \cap A = \emptyset$. ↯

This is a contradiction to every nbhd of x intersecting $A \setminus \{x\}$.

So $x \in \bar{A}$, $\bar{A} \cap V = \emptyset \Rightarrow \bar{A} \not\subseteq U \cup V$.

Therefore, if A is connected, then $\bar{A}$ is connected. □

2. If A is connected, then its interior $Int(A)$ is connected.

Pf: Let $A = \underbrace{\{(x,y) \in \mathbb{R}^2 : x < 0\}}_{\text{left-half plane}} \cup \underbrace{\{(x,y) \in \mathbb{R}^2 : x > 0\}}_{\text{right-half plane}} \cup \underbrace{\{(0,0)\}}_{\text{origin}}$.

Observe that A is connected, and that

$Int(A) = \underbrace{\{(x,y) \in \mathbb{R}^2 : x < 0\}}_{\text{left-half plane}} \cup \underbrace{\{(x,y) \in \mathbb{R}^2 : x > 0\}}_{\text{right-half plane}}$ is disconnected.

There is no open ball around $(0,0)$, say $B_{(0,0)}(r)$, where $r > 0$, s.t. $B_{(0,0)}(r) \subseteq A$. Therefore, $(0,0) \notin Int(A)$, so $Int(A)$ is $\mathbb{R}^2$ without the y-axis. Thus, $Int(A)$ is not connected. □

Continued...

② Define the equivalence relation on $\mathbb{R}$ such that $x \sim y$ if $x-y$ is rational. Let $\mathbb{R}/\sim$ be the quotient space with the quotient topology. Show that $\mathbb{R}/\sim$ is not Hausdorff.

Pf: Let $q: \mathbb{R} \to \mathbb{R}/\sim$ be the quotient map.

Observe that $[0]$ and $[\pi]$ are two distinct points in $\mathbb{R}/\sim$ since $\pi - 0 = \pi \notin \mathbb{Q}$.

We WTS that every nbhd of $[0]$ intersects every nbhd of $[\pi]$.

Let U be an open nbhd of $[0]$.

Since q is continuous and U is open in $\mathbb{R}/\sim$, $q^{-1}(U)$ is open in $\mathbb{R}$.

We know that $0 \in q^{-1}(U)$.

Since $q^{-1}(U)$ is open, we have that $0 \in (-\varepsilon, \varepsilon) \subseteq \underbrace{q^{-1}(U)}_{\text{this is saturated and contains an interval.}}$ for some $\varepsilon > 0$.

Let $x \in \mathbb{R}$ be arbitrary. Choose $r \in (x-\varepsilon, x+\varepsilon)$, $r \in \mathbb{Q}$.

So $x - r \in (-\varepsilon, \varepsilon) \Rightarrow x - r \in q^{-1}(U)$, so $x - r \sim x$.

Since $x - r \sim x$ and $q^{-1}(U)$ is saturated and $x - r \in q^{-1}(U)$, we have that $x \in q^{-1}(U)$.

Since x was arbitrary in $\mathbb{R}$, we have that $\mathbb{R} = q^{-1}(U)$.

Taking q of both sides, we get $\underset{\mathbb{R}/\sim}{\underset{\parallel}{q(\mathbb{R})}} = q(q^{-1}(U)) = U$ since q is surj.

So we have shown that $U = \mathbb{R}/\sim$, so $U \cap V \neq \emptyset$ for any open, nonempty $V \subseteq \mathbb{R}/\sim$.

Therefore, we conclude that $\mathbb{R}/\sim$ is not Hausdorff.

□

...nued...

(3) Let X, Y be topological spaces. Assume that Y is Hausdorff. Let $f, g: X \to Y$ be continuous functions. Suppose that there exists a dense subset D of X such that $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$.

Pf: Let $A = \{x \in X : f(x) = g(x)\}$

We want to show that A is closed, so we will do this by showing that $X \setminus A = \{x \in X : f(x) \neq g(x)\}$ is open.

Let $x \in X \setminus A$. Then $f(x) \in Y$.

Take $g(x) \in Y$. Since $x \in X \setminus A$, we know that $f(x) \neq g(x)$.

Since Y is Hausdorff, there exist open nbhds U of $f(x)$ and V of $g(x)$ such that $U \cap V = \emptyset$.

Since f, g are cts and U, V are open in Y, we have that $f^{-1}(U)$ and $g^{-1}(V)$ are open in X.

Observe that $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$.

Let $W := f^{-1}(U) \cap g^{-1}(V)$.

W is open (finite intersection of open sets is open) and W is nonempty ($x \in W$) and a nbhd of x.

We WTS that $W \cap A = \emptyset$:

Let $y \in W$. Then $f(y) \in U$ because $W \subseteq f^{-1}(U)$.

Since $y \in W$, we also have $g(y) \in V$ because $W \subseteq g^{-1}(V)$.

But $U \cap V = \emptyset$, so $f(y) \neq g(y)$ for all $y \in W$.

Therefore, $W \cap A = \emptyset$ by definition of A.

So $x \in \underbrace{W}_{\text{open}} \subseteq X \setminus A$.

Therefore, $X \setminus A$ is open $\Rightarrow$ A is closed.

So we have that A is a closed set s.t. $D \subseteq A \subseteq X$.

Since D is dense in X, we get that $A = X$.

Thus, $f(x) = g(x)$ $\forall x \in X$. $\square$

continued...

(4) A topological space X is said to be contractible if the identity map $Id_X: X \to X$ is null-homotopic, i.e., homotopic to a constant map.

1. Show that any convex subset of $\mathbb{R}^n$ is contractible.

Pf: Let $A \subseteq \mathbb{R}^n$ be convex.

We WTS that $Id_A: A \to A$ is homotopic to a constant map, C_{x_0}.

Let $H: [0,1] \times A \to X$ be defined by $H(t,x) = (1-t)x + tx_0$.

First we will check that H is a homotopy: (H is clearly cts)

$H(0,x) = Id_A(x) = x,$

$H(1,x) = C_{x_0}(x) = x_0.$

Therefore, H is a homotopy.

Since A is convex, $H(t,x) \in A$ for all $t \in [0,1]$, $x \in A$.

Thus, A is contractible. $\square$

2. Let Y be a topological space. Show that if X is contractible, then any map $f: X \to Y$ is null-homotopic.

Pf: Since X is contractible, $Id_X: X \to X$ is homotopic to a constant map, C_{x_0}.

Let $H: [0,1] \times X \to X$ such that $H(0,x) = Id_X(x) = x,$
$H(1,x) = C_{x_0}(x) = x_0.$

Then define $\tilde{H}: [0,1] \times X \to Y$ by $\tilde{H}(t,x) = (f \circ H)(t,x)$.

$\tilde{H}$ is cts since it is the comp. of cts. fns.

We will check that $\tilde{H}$ is a homotopy:

$\tilde{H}(0,x) = f(H(0,x)) = f(x)$

$\tilde{H}(1,x) = f(H(1,x)) = \underbrace{f(x_0)}_{\text{this is some constant.}}$

Therefore, $\tilde{H}$ is a homotopy.

Thus, f is null-homotopic to a constant map.

$\square$

...tinued...

⑤ Let E, X be topological spaces. Assume that E is connected. Let $q: E \to X$ be a covering map with $q^{-1}(x)$ finite and nonempty for all $x \in X$. Show that E is compact if and only if X is compact.

Pf: · Suppose E is compact.

Covering maps are continuous and surjective.

So we have that $q(E) = X$ and since q cts, E cpt $\Rightarrow q(E) = X$ is cpt.

(the cts image of a compact set is compact).

· Suppose X is compact.

Let $\{U_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of E ($U_\alpha \subseteq E$ open).

Claim: For each $x \in X$, $\exists$ a nbhd V_x of x s.t. $q^{-1}(V_x)$ can be covered by finitely many U_α's.

Assuming the claim, we can write $X = \bigcup_{x \in X} V_x = \bigcup_{i=1}^{n} V_{x_i}$ (b/c X is cpt).

Then $E = q^{-1}(X) = \bigcup_{i=1}^{n} q^{-1}(V_{x_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_j^i$, where each $U_j^i \in \{U_\alpha : \alpha \in I\}$

(Let $U_1^i, \dots, U_{n_i}^i$ is a finite collection of U_α's that cover $q^{-1}(V_{x_i})$)

So we have $E \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_j^i$ is a finite union $\Rightarrow$ finite subcover of E.

Therefore, E is compact. $\square$

Proof of claim: We have $\{U_\alpha\}_{\alpha \in I}$ an arbitrary open cover of E, $x \in X$.

We want a nbhd V of x s.t. $q^{-1}(V)$ can be covered by finitely many U_α's.

Since $q^{-1}(x)$ is finite, write $q^{-1}(x) = \{e_1, \dots, e_n\}$.

Let U_i be a member of the $\{U_\alpha\}$ cover that contains e_i for $i = 1, \dots, n$.

Let W be any evenly covered nbhd of x and write $q^{-1}(W) = \bigcup_{i=1}^{n} W_i$.

We can relabel these slices if needed so that $e_i \in W_i$.

Then $e_i \in U_i \cap W_i$ for each i.

Since q is open, the sets $q(U_i \cap W_i)$ are open in X and contain x for each i.

Set $V = \bigcap_{i=1}^{n} q(U_i \cap W_i)$.

We claim that $q^{-1}(V) \subseteq \bigcup_{i=1}^{n} U_i$. Let $y \in V$ arbitrary.

By definition of V, there exists $y_i \in U_i \cap W_i$ s.t. $q(y_i) = y$ for $i = 1, \dots, n$.

Disjointness of the W_i guarantees the y_i's are distinct.

continued...

We know the fiber $q^{-1}(y)$ intersects each W_i exactly once (recall that $y \in W$ and the W_i's are the slices of $q^{-1}(W)$).

Hence the fiber $q^{-1}(y)$ contains exactly n points, so we have shown $q^{-1}(y) = \{y_1, \dots, y_n\} \subset \bigcup_{i=1}^{n} U_i$.

Since $y \in V$ was aribrary, we conclude $q^{-1}(V) \subseteq \bigcup_{i=1}^{n} U_i$. $\square$

...nued...

(6) Let $n \geq 3$. Suppose that M is a connected n-dimensional manifold, and $p \in M$. Show that the inclusion $M \setminus \{p\} \hookrightarrow M$ indues an isomorphism between their fundamental groups $\pi_1(M \setminus \{p\}) \cong \pi_1(M)$.

Pf Note that connected manifolds are path-connected. Therefore, $\pi_1(M)$ does not depend on the base point.

Let $p, q \in M$ be distinct. It suffices to show that every loop at q in M is homotopic to a loop at q that does not pass through p.

Let $[f] \in \pi_1(M, q)$.

Let U be a nbhd of p that is homeomorphic to the open unit ball in $\mathbb{R}^n$.

Let $V = M \setminus \{p\}$. Since $U \cap V = M$, we must have that $\{f^{-1}(U), f^{-1}(V)\}$ is an open cover of $[0,1]$.