...st 2016

① Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ with the topology given by the basis $\mathcal{B} = \{[a,b) : a<b \text{ and } a,b \in \mathbb{Q}\}$. Determine the closure of the following subsets in $\mathbb{R}_\ell$.

(i) $(1,\sqrt{2})$.

Pf: We know that $(1,\sqrt{2}) \subseteq \overline{(1,\sqrt{2})}$.

Recall that x is a limit point of A if every deleted nbhd of x intersects A. (If $x \in \bar{A}$, then every nbhd of x intersects A).

If $x<1$, then a nbhd of x is $[a,1)$ where $a \in \mathbb{Q}$, $a \leq x < 1$.

We have that $[a,1) \cap (1,\sqrt{2}) = \emptyset$.

Therefore, every $x<1$ is not in $\overline{(1,\sqrt{2})}$.

If $x > \sqrt{2}$, then a nbhd of x is $[a,b)$ where $a,b \in \mathbb{Q}$, $\sqrt{2} \leq a < x < b$.

We have that $(1,\sqrt{2}) \cap [a,b) = \emptyset$.

Therefore, every $x > \sqrt{2}$ is not in $\overline{(1,\sqrt{2})}$.

Now we want to check if 1 and $\sqrt{2}$ are in $\overline{(1,\sqrt{2})}$.

If $x=1$, then a nbhd of x is $[a,b)$ where $a,b \in \mathbb{Q}$, $a \leq 1 < b$.

We have that $[a,b) \cap (1,\sqrt{2}) = (1,c) \neq \emptyset$ $\quad (c = \min\{1,\sqrt{2}\})$

$\hookrightarrow c \in \mathbb{Q}$ s.t. $1 < c \leq \sqrt{2}$ or b (whichever is smaller)

$(\sqrt{2} \notin \mathbb{Q})$

Therefore, $x=1$ is in $\overline{(1,\sqrt{2})}$.

If $x = \sqrt{2}$, then a nbhd of x is $[a,b)$ where $a,b \in \mathbb{Q}$, $a < \sqrt{2} < b$.

We have that $(1,\sqrt{2}) \cap [a,b) = (c,\sqrt{2}) \neq \emptyset$

$\hookrightarrow c \in \mathbb{Q}$ s.t. $1 \leq c \leq a$

Therefore, $x = \sqrt{2}$ is in $\overline{(1,\sqrt{2})}$.

Thus, we conclude that $\overline{(1,\sqrt{2})} = [1,\sqrt{2}]$ in $\mathbb{R}_\ell$.

□

continued...

(ii) $(\sqrt{2}, 3)$

Pf: We know that $(\sqrt{2}, 3) \subseteq \overline{(\sqrt{2}, 3)}$.

We will use the same approach as in part (i). $(\sqrt{2} \notin \mathbb{Q})$

If $x < \sqrt{2}$, then a nbhd of x is $[a, b)$, where $a, b \in \mathbb{Q}$, $a \leq x < b < \sqrt{2}$.

We have that $[a, b) \cap (\sqrt{2}, 3) = \emptyset$.

Therefore, every $x < \sqrt{2}$ is not in $\overline{(\sqrt{2}, 3)}$.

If $x > 3$, then a nbhd of x is $[3, b)$ where $b \in \mathbb{Q}$, $3 < x < b$.

We have that $(\sqrt{2}, 3) \cap [3, b) = \emptyset$.

Therefore, every $x > 3$ is not in $\overline{(\sqrt{2}, 3)}$.

Now we want to check if $\sqrt{2}$ and 3 are in $\overline{(\sqrt{2}, 3)}$.

If $x = \sqrt{2}$, then a nbhd of x is $[a, b)$ where $a, b \in \mathbb{Q}$, $a < \sqrt{2} < b$ $(\sqrt{2} \notin \mathbb{Q})$.

We have that $[a, b) \cap (\sqrt{2}, 3) = (\sqrt{2}, c)$

$\hookrightarrow c \in \mathbb{Q}$ s.t. $\sqrt{2} < c < b$ or 3 (whichever is smaller)

Therefore, $x = \sqrt{2}$ is in $\overline{(\sqrt{2}, 3)}$.

If $x = 3$, then a nbhd of x is $[a, b)$ where $a, b \in \mathbb{Q}$, $a \leq 3 < b$.

We have that $(\sqrt{2}, 3) \cap [a, b) = \emptyset$ if $a = 3$.

Therefore, $x = 3$ is not in $\overline{(\sqrt{2}, 3)}$.

Thus, we conclude that $\overline{(\sqrt{2}, 3)} = [\sqrt{2}, 3)$ in $\mathbb{R}_\ell$.

$\square$

Continued...

③ Let $f: X \to Y$ be a continuous and injective map between topological spaces X and Y. Prove that if X is compact and Y is Hausdorff, then f is an embedding.

Pf: Since f is continuous and injective, it suffices to prove that f is closed.

Let $K \subseteq X$ be closed.

Closed subsets of compact spaces are compact.

Since K closed, X compact, $K \subseteq X \Rightarrow K$ is compact.

The continuous image of a compact set is compact.

Since f continuous, K compact $\Rightarrow f(K)$ is compact, $f(K) \subseteq Y$.

Compact subsets of Hausdorff spaces are closed.

Since $f(K)$ compact, Y Hausdorff, $f(K) \subseteq Y \Rightarrow f(K)$ is closed.

Therefore, if $K \subseteq X$ is closed, then $f(K) \subseteq Y$ is closed.

Thus, f is closed.

We conclude that if X compact, Y Hausdorff, then f is an embedding.

$\square$

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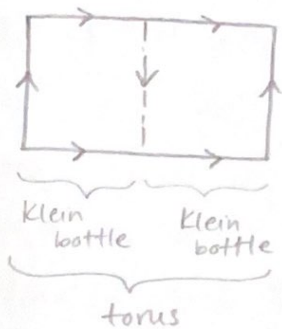
(4) Show that there is a two-sheeted covering of the Klein bottle by the torus $\mathbb{T}^2$.

Pf: Observe that if we take the torus (polygon rep.) and draw a line through the middle as follows, we have two Klein bottles:



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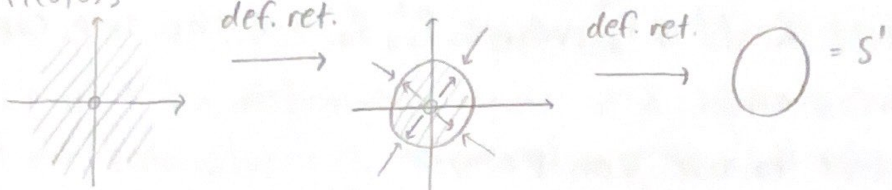
(ii) Use part (i) to give the fundamental group of each of the following spaces.

(1) $\mathbb{R}^2$ with the origin removed.

Pf: $\mathbb{R}^2 \setminus \{(0,0)\}$

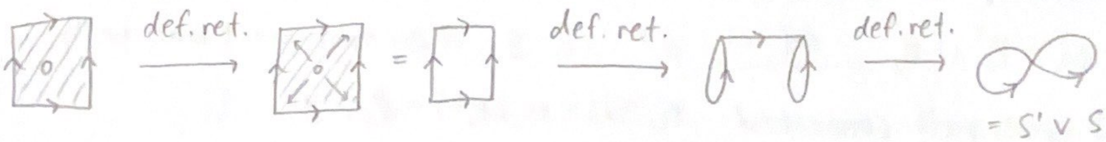


$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \pi_1(S^1) = \mathbb{Z}$.

□

(2) The torus $S^1 \times S^1$ with one point removed.

Pf:



Let $U =$  $\xrightarrow{\text{def. ret.}}$  $= S^1$

U is path-conn., open. $\pi_1(U) = \pi_1(S^1) = \mathbb{Z}$

Let $V =$  $\xrightarrow{\text{def. ret.}}$  $= S^1$

V is path-conn., open. $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$

Observe that $S^1 \vee S^1 = U \cup V$

$U \cap V =$  $\xrightarrow{\text{def. ret.}} \cdot$

$U \cap V$ is path-conn., and nonempty. $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply conn., we can use the following version of Van-Kampen $\pi_1(S^1 \vee S^1) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z}$.

By part (i), $\pi(T^2) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

□

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(6) Let X be a topological space obtained by taking two copies of real projective plane P^2 and identifying a single point p in one copy with a single point q in another copy. Determine the fundamental group of X.

Pf: We have that $X = P_1^2 \vee P_2^2$ where P_1^2, P_2^2 are the two copies of the real projective plane.

We would like to use Van-Kampen.

Let x be the wedge point of X.

Every point of P_1^2 and P_2^2 is an interior point, except the wedge point, so let $B_1 \subseteq P_1^2$ and $B_2 \subseteq P_2^2$ be open nbhds of x that are homeomorphic to the unit ball in $\mathbb{R}^2$

Let $U = P_1^2 \cup B_2 \xrightarrow{\text{def. ret.}} P_1^2$ b/c B_2 is an open unit ball in P_2

U is open, path-connected. $\pi_1(U) = \pi_1(P_1^2) = \mathbb{Z}/2\mathbb{Z}$.

Let $V = P_2^2 \cup B_1 \xrightarrow{\text{def. ret.}} P_2^2$ b/c B_1 is an open unit ball in P_1

V is open, path-connected. $\pi_1(V) = \pi_1(P_2^2) = \mathbb{Z}/2\mathbb{Z}$.

Observe that $X = U \cup V$.

$U \cap V = (P_1^2 \cup B_2) \cap (P_2^2 \cup B_1) = B_1 \cup B_2$

$U \cap V$ is nonempty, open, and path-connected. $\pi_1(U \cap V) = \pi_1(B_1 \cup B_2) = 0$.

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen:
$$\begin{aligned}\pi_1(X) &= \pi_1(P_1^2 \vee P_2^2) \\ &= \pi_1(U \cup V) \\ &= \pi_1(U) * \pi_1(V) \\ &= \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Therefore, $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. $\square$