...st 2017

1) Let $X = \mathbb{N} = \{1, 2, 3, \dots\}$ and equip X with the topology $\tau = \{U \subseteq X : (2n-1) \in U \Rightarrow 2n \in U\}$. That is, $U \in \tau$ if and only if every odd number that is contained in U has its successor also contained in U.

(a) Prove that (X, τ) is not compact, but it is locally compact (i.e., any point has a compact nbhd.).

Pf: For each $n \in \mathbb{N}$, the set $U_n = \{2n-1, 2n\}$ is open in X.

Notice that $X = \bigcup_{n=1}^{\infty} U_n$, so the family $\{U_n : n \in \mathbb{N}\}$ is a disjoint open cover of X.

Since each integer belongs to exactly one U_n, we have that this cover is minimal, i.e., if one member of the cover is removed, the resulting collection no longer covers the space.

Therefore, $\{U_n : n \in \mathbb{N}\}$ does not admit any proper subcover, let alone a finite one.

Thus, X is not compact.

To see that the space is locally compact, let $m \in X$ be arbitrary, and set $n = \text{ceil}\left(\frac{m}{2}\right)$.

Then $m \in U_n = \{2n-1, 2n\}$ as defined above.

The set U_n is open and compact, so it is a compact nbhd of m, as needed. $\square$

continued...

(b) Determine (with proof) the connected components of (X, τ).

Pf: Let $U_n = \{2n-1, 2n\}$.

We claim that each U_n is a connected component of X.

Since the U_n's cover the space, this means that there are no other connected components.

- First, we show that U_n is connected.
  The open subsets of U_n are precisely $\{2n\}$, $\emptyset$, and U_n.
  It is clear that U_n cannot be written as the disjoint union of any pair of these, so U_n is connected.

- Next, we will demonstrate that U_n is a maximal connected subset of X.
  It will be helpful to first show that U_n is clopen in X.
  This is immediate since we have already shown U_n is open, and the complement of U_n is $\{1, 2, \dots, 2n-2, 2n+1, \dots\}$.
  The only even number missing from this set is $2n$, but its predecessor is also missing, so the set is open.

- Now, if $U_n \subsetneq A \subset X$, we see that $A \cap U_n = U_n$ is both closed and open in the subspace topology.
  Since A properly contains U_n and U_n is nonempty, this demonstrates that A is disconnected.
  Thus, U_n is a maximal connected subset, and we are done. $\square$

...ed...

Let X, Y be topological spaces, D a dense subset of X and $f, g: X \to Y$ continuous maps such that $f(x) = g(x)$, for all $x \in D$. Show that if Y is Hausdorff, then $f = g$ on X.

Pf: Recall that $D \subseteq X$ dense means $\overline{D} = X$.

Let $A = \{x \in X : f(x) = g(x)\}$. We WTS that A is closed.

We will do so by showing that $X \setminus A = \{x \in X : f(x) \neq g(x)\}$ is open.

Let $x \in X \setminus A$. Then $f(x), g(x) \in Y$ s.t. $f(x) \neq g(x)$.

Since Y is Hausdorff and $f(x) \neq g(x)$, we have that $\exists$ open nbhds U of $f(x)$ and V of $g(x)$ s.t. $U \cap V = \emptyset$.

Since f, g are continuous and $U, V \subseteq Y$ are open, we have that $f^{-1}(U)$ and $g^{-1}(V)$ are open in X.

We know that $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$.

Let $W := f^{-1}(U) \cap g^{-1}(V)$.

W is open b/c the finite intersection of open sets is open, and W is a nonempty nbhd of x ($x \in W$).

We WTS that $W \cap A = \emptyset$:

Let $y \in f^{-1}(U) \cap g^{-1}(V) = W$.

$$\text{Then } \left.\begin{array}{l} y \in f^{-1}(U) \Rightarrow f(y) \in U \\ y \in g^{-1}(V) \Rightarrow g(y) \in V \end{array}\right\} U \cap V = \emptyset, \text{ so } f(y) \neq g(y) \ \forall y \in W.$$

Therefore, $W \cap A = \emptyset$.

Thus, we have $x \in \overset{\text{open}}{W} \subseteq X \setminus A$, so $X \setminus A$ is open $\Rightarrow A$ is closed.

A is a closed subset s.t. $D \subseteq A \subseteq X$.

Since D is dense in X, we get that $A = X$.

Therefore, $f(x) = g(x)$ $\forall x \in X$.

$\square$

continued...

(3) Prove that there exists no one-to-one continuous map from $\mathbb{R}^n$ to $\mathbb{R}$ for $n > 1$. Hint: How would such a map act on the unit sphere?

Pf: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a one-to-one continuous map.

Observe that $S^2 \subseteq \mathbb{R}^3$ and S^2 is path-connected and compact.

The continuous image of a path-conn. space is path-connected.

Since f is cts and S^2 is path-conn., we have that $f(S^2) \subseteq \mathbb{R}$ must be path-connected.

The path-connected subsets of $\mathbb{R}$ are intervals, so $f(S^2)$ must be some interval in $\mathbb{R}$.

The continuous image of a compact space is compact.

Since f is cts and S^2 is cpt, we have that $f(S^2) \subseteq \mathbb{R}$ is compact.

Therefore, $f(S^2) = [a, b]$ some closed and bounded interval in $\mathbb{R}$.

Observe that $S^2 \setminus \{\text{a point}\}$ is still connected, and that $[a,b] \setminus \{\underbrace{\text{a point}}_{= c}\}$ is equal to $[a,c) \cup (c,b]$ which is disconnected.

Therefore, there exists no one-to-one continuous map from $\mathbb{R}^n$ to $\mathbb{R}$ for $n > 1$.

$\square$

...ed.

Let $\mathbb{D}$ be the closed unit disk, and S^1 the unit circle.

(a) Prove that there is no retraction $r: \mathbb{D} \to S^1$.

Pf: Assume that such a retraction $r: \mathbb{D} \to S^1$ exists.

Then we have that the induced homomorphism $r_*: \pi_1(\mathbb{D}) \to \pi_1(S^1)$ is surjective.

Observe that $\pi_1(\mathbb{D}) = 0$ and $\pi_1(S^1) = \mathbb{Z}$.

This contradicts r_* being surjective.

Therefore, there is no retraction $r: \mathbb{D} \to S^1$. $\square$

(b) Prove that every continuous map $f: \mathbb{D} \to \mathbb{D}$ has a fixed point.

Pf: Assume $f: \mathbb{D} \to \mathbb{D}$ is a continuous map with no fixed point.

For each $x \in \mathbb{D}$, define $r(x)$ in the following way:

$r(x)$ is the intersection of the ray from $f(x)$ to x with S^1, where this ray does not include the endpoint $f(x)$.

Since f has no fixed points, this map is well-defined.

This map is also continuous.

It is also clear that $r: \mathbb{D} \to S^1$ fixes S^1, so it is a retraction of $\mathbb{D}$ onto S^1.

This induces $r_*: \pi_1(\mathbb{D}) \to \pi_1(S^1)$ in the usual way.

If we let $j: S^1 \to \mathbb{D}$ denote inclusion, then $r \circ j$ is the identity of S^1, so $(r \circ j)_* = r_* \circ j_*$ is the identity automorphism of $\pi_1(S^1)$.

This implies r_* is surjective.

But since $\pi_1(\mathbb{D})$ is trivial, r_* is trivial.

Since $\pi_1(S^1)$ is nontrivial, r_* cannot be trivial and surjective. $\lightning$

This is a contradiction.

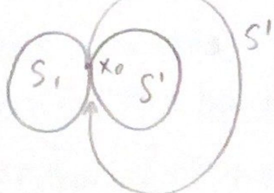
Therefore, we conclude that every continuous map $f: \mathbb{D} \to \mathbb{D}$ has a fixed point.

$\square$

Continued...

(5) Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other.

Pf: Recall that the torus is homeomorphic to $S^1 \times S^1$.
Thus, when identifying a circle $S^1 \times \{x_0\}$ on one torus to a circle $S^1 \times \{x_0\}$ on the other, we have that this space is the same as identifying a single point between the first S^1 on each torus and the entire second S^1 on each torus.



Since identifying a single point is the wedge product, we have that this space is homotopic to $(S^1 \vee S^1) \times S^1$.
Since $\pi_1(S^1) = \mathbb{Z}$, we have that

$$\begin{aligned}\pi_1((S^1 \vee S^1) \times S^1) &= \pi_1(S^1 \vee S^1) \times \pi_1(S^1) \\ &= (\pi_1(S^1) * \pi_1(S^1)) \times \pi_1(S^1) \\ &= (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}.\end{aligned}$$

□

ued.

Let K be the Klein bottle and T the two-dimensional torus. Prove or disprove:

(a) There is a covering map from K to T.

Pf. Assume that there exists a covering map $q: K \to T$.

Then the induced homomorphism $q_*: \pi_1(K) \to \pi_1(T)$ is injective.

Observe that $\pi_1(K) = \langle a, b : abab^{-1} = 1 \rangle$

$= \langle a, b : aba = b \rangle$ nonabelian

and $\pi_1(T) = \pi_1(S^1 \times S^1)$

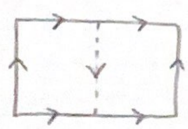
$= \pi_1(S^1) \times \pi_1(S^1)$

$= \mathbb{Z} \times \mathbb{Z}$ abelian.

$\square$

(b) There is a covering map from T to K.

Pf: Observe that if we take a torus and draw a line down the middle as follows



. we have two klein bottles.

This is a covering.

Therefore, there is a covering from T to K.

$\square$