ust 2018

1) Let Γ be a subset in a compact topological space such that every point of Γ is an isolated point of Γ. Is Γ necessarily a finite set? Prove your assertion.

Pf: Γ is not necessarily a finite set.

Consider the compact space $[0,1]$ of $\mathbb{R}$.

Let $\Gamma = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Observe that every point of Γ is an isolated point:

For each point $\frac{1}{n} \in \Gamma$, $\exists$ an open nbhd $B_\varepsilon(\frac{1}{n})$, where $\varepsilon = \frac{(\frac{1}{n} - \frac{1}{n+1})}{2}$.

Then $B_\varepsilon(\frac{1}{n}) \cap \Gamma = \{\frac{1}{n}\}$, so each $\frac{1}{n} \in \Gamma$ is an isolated point of Γ.

Also, notice that Γ is not finite.

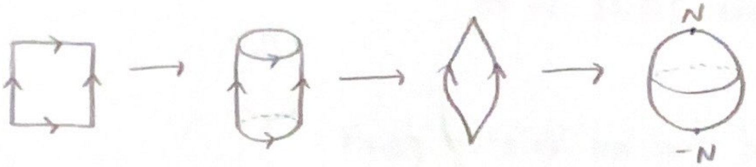
Therefore, a subset Γ, of a compact space, s.t. every point of Γ is an isolated point of Γ, need not necessarily be a finite set. $\square$

Continued...

② Compute the fundamental group of the quotient space $(S^1 \times S^1)/(S^1 \times \{x\})$, where x is a point in S^1.

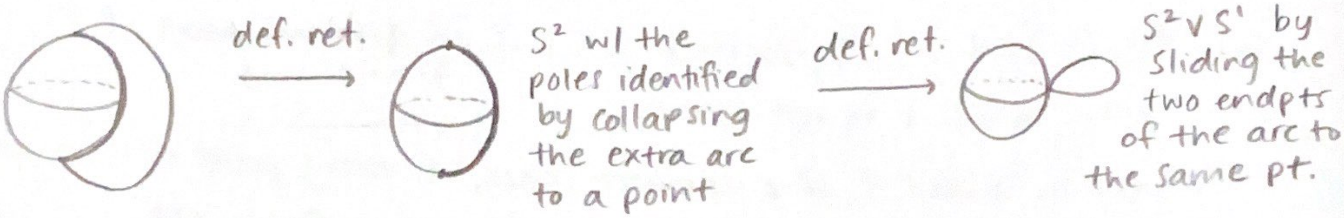
Pf: Let $X = (S^1 \times S^1)/(S^1 \times \{x\})$.

Here is a construction of the pinched torus:



This is S^2 with two points (N and $-N$) identified togeter

X is homeomorphic to the sphere with two points identified



So we have that $\pi_1(X) = \pi_1(S^2 \vee S^1)$.

Each space, S^2 and S^1, is locally Euclidean, so the wedge point has a nbhd in each space that deformation retracts to the wedge point (itself).

Therefore, we can use the following version of Van-Kampen:

$$\pi_1(S^2 \vee S^1) = \pi_1(S^2) * \pi_1(S^1) = 0 * \mathbb{Z} = \mathbb{Z}.$$

Thus, we conclude that $\pi_1(X) = \mathbb{Z}$. $\square$

...ued.

Let $\mathcal{Z}$ be the topology on $\mathbb{R}^2$ such that every nonempty open set of $\mathcal{Z}$ is of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many point}\}$.

(i) Is $(\mathbb{R}^2, \mathcal{Z})$ Hausdorff? Prove your assertion.

Pf: Let $X = (\mathbb{R}^2, \mathcal{Z})$. We WTS that X is not Hausdorff, i.e., there exist two distinct points $x, y \in X$ s.t. every open nbhd of x intersects every open nbhd of y.

Consider the distinct points $(0,0)$ and $(0,1)$ in X.

Let U be any open nbhd of $(0,0)$ and V be any open nbhd of $(0,1)$.

Then U and V must be of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many points}\}$.

So $X \setminus U$ and $X \setminus V$ must be finite.

If $U \cap V = \emptyset$, then $V \subseteq \underbrace{X \setminus U}_{\text{finite}} \Rightarrow V$ is finite

But $\mathbb{R}^2 = \underset{\text{finite}}{V} \cup \underset{\text{finite}}{V^c} \Rightarrow \mathbb{R}^2$ is finite. $\lightning$

Therefore, $U \cap V \neq \emptyset$.

Thus, $(\mathbb{R}^2, \mathcal{Z})$ is not Hausdorff. $\square$

(ii) Is $(\mathbb{R}^2, \mathcal{Z})$ first countable? Prove your assertion.

Pf: We WTS that X is not first countable. $(X = (\mathbb{R}^2, \mathcal{Z}))$

We will do this by showing that 0 does not have a countable local basis.

Assume $\{U_n : n \in \mathbb{N}\}$ is any countable collection of nbhds of 0.

To prove $\{U_n\}$ is not a local basis, we need to find a nbhd V of 0 s.t. V does not contain any U_n.

Let $V = \bigcup_{n=1}^{\infty} U_n^c$. Then V is a countable union of finite sets, so V is countable. $V \neq \mathbb{R}^2$.

Since X is uncountable, we may consider some nonzero $x \in X \setminus V$ because $X \setminus V$ is an infinite set.

Then the set $X \setminus \{x\}$ is an open nbhd of 0 that does not contain any of the U_n's, since $x \in U_n$ for all n.

Hence, $\{U_n\}_{n=1}^{\infty}$ is not a local basis at 0.

Therefore, it follows that X is not first countable. $\square$

continued...

(4) Show that every continuous map from $\mathbb{RP}^2$ to S^1 is homotopic to a constant map

Pf: Let $f: \mathbb{RP}^2 \to S^1$ be a continuous map.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

We would like to use the general lifting lemma to show that there exists a lift $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}$.

Observe that $\mathbb{RP}^2$ is path-connected because it is the cts image of S^2, a path-conn. space.

Observe that $\mathbb{RP}^2$ is locally path-connected because $q: S^2 \to \mathbb{RP}^2$ is a local homeomorphism.

Let $p: \mathbb{R} \to S^1$ (the exp. map) be a covering map.

It remains to show that $f_*(\pi_1(\mathbb{RP}^2)) \subseteq p_*(\pi_1(\mathbb{R}))$.

Since $\pi_1(\mathbb{R}) = 0$, it suffices to show that $f_*(\pi_1(\mathbb{RP}^2)) = 0$.

Notice that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ is finite, so $f_*(\pi_1(\mathbb{RP}^2))$ is finite.

We have that $f_*(\pi_1(\mathbb{RP}^2)) \leq \pi_1(S^1) = \mathbb{Z}$.

The only finite subgroup of $\mathbb{Z}$ is 0.

So $f_*(\pi_1(\mathbb{RP}^2)) = 0 \subseteq p_*(\pi_1(\mathbb{R}))$ ✓.

Therefore, by the general lifting lemma, we have that $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}$ is a lift.

Recall that any cts fn. into a contractible space is null-homotopic.

Since $\tilde{f}$ is cts and $\mathbb{R}$ is contractible, we have that $\tilde{f}$ is null-homotopic.

Since $\tilde{f}$ is null-homotopic, we have that f is null-homotopic (if H is a homotopy between $\tilde{f}$ and a constant map, then $p \circ H$ is a homotopy between f and a constant map).

Therefore, f is null-homotopic.

Thus, we conclude that every cts map from $\mathbb{RP}^2$ to S^1 is homotopic to a constant map.

$\square$

...ued ..

Let M be the quotient space of $\mathbb{R}^3 \setminus \{0\}$ obtained by identifying the points (x, y, z) with $(2^m x, 2^m y, 2^m z)$ for any integer m. Is M homeomorphic to $S^2 \times S^1$? Prove your assertion.

Pf. We may view $\mathbb{R}^3 \setminus \{0\}$ as $S^2 \times (0, \infty)$.

With this in mind, we can write $M = (S^2 \times (0,\infty))/\sim$, where $\sim$ is the equiv. rel. from the problem.