...st 2019

Let X be a topological space and A, B be subsets of X. Prove or disprove the following set equalities.

(a) $\overline{X \setminus (A \cup B)} = X \setminus (\text{Int}(A) \cup \text{Int}(B))$.

Pf: False.

Let $X = \mathbb{R}$. Consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$.

Then $\text{Int}(A) = \emptyset$ and $\bar{A} = \mathbb{R}$. Similarly, $\text{Int}(B) = \emptyset$ and $\bar{B} = \mathbb{R}$.

We can write $\overline{X \setminus (A \cup B)} = X \setminus (\text{Int}(A) \cup \text{Int}(B))$ as $\overline{(X \setminus A) \cap (X \setminus B)} = (X \setminus \text{Int}(A)) \cap (X \setminus \text{Int}(B)) \Rightarrow \overline{A^c \cap B^c} = (\text{Int}(A))^c \cap (\text{Int}(B))^c$.

Observe that $\underbrace{(\text{Int}(A))^c}_{\mathbb{R} \setminus \emptyset} \cap \underbrace{(\text{Int}(B))^c}_{\mathbb{R} \setminus \emptyset} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

Observe that $\overline{A^c \cap B^c} = \overline{(\mathbb{R} \setminus \mathbb{Q}) \cap \mathbb{Q}} = \emptyset$.

But $\mathbb{R} \neq \emptyset$ ↯ Contradiction.

Therefore, $\overline{X \setminus (A \cup B)} \neq X \setminus (\text{Int}(A) \cup \text{Int}(B))$ $\forall A, B \subseteq X$. □

(b) $\text{Int}(X \setminus (A \cup B)) = X \setminus (\bar{A} \cup \bar{B})$.

Pf: True.

First, we will rewrite $\text{Int}(X \setminus (A \cup B)) = X \setminus (\bar{A} \cup \bar{B})$ as $\text{Int}(A^c \cap B^c) = \bar{A}^c \cap \bar{B}^c$.

- Let $x \in \text{Int}(A^c \cap B^c)$. Then $x \in U$ for some $U \subseteq A^c \cap B^c$, U open.
  $U \subseteq A^c$ and $U \subseteq B^c$, so $x \in \text{Int}(A^c) \cap \text{Int}(B^c)$.
  So $x \in \text{Int}(A^c)$ since $x \in U \subseteq A^c$ where U is open.
  $x \in \bar{A}$ iff every nbhd of x intersects A.
  But $x \in U$ where U is open and disjoint from A, so $x \notin \bar{A}$. $\Rightarrow x \in \bar{A}^c$.
  Similarly, $x \in \text{Int}(B^c)$ since $x \in U \subseteq B^c$ where U is open.
  But $x \in U$, where U is open and disjoint from B, so $x \notin \bar{B} \Rightarrow x \in \bar{B}^c$.
  Therefore, we have that $x \in \bar{A}^c \cap \bar{B}^c$. So $\text{Int}(A^c \cap B^c) \subseteq \bar{A}^c \cap \bar{B}^c$.

- Now let $x \in \bar{A}^c \cap \bar{B}^c$.
  $x \notin \bar{A} \Rightarrow \exists\, U$ open with $x \in U$ and $U \cap A = \emptyset \Rightarrow U \subseteq A^c$
  $x \notin \bar{B} \Rightarrow \exists\, V$ open with $x \in V$ and $V \cap B = \emptyset \Rightarrow V \subseteq B^c$
  So we have $x \in \underbrace{U \cap V}_{\text{open b/c finite intersection of open sets is open}} \subseteq A^c \cap B^c \Rightarrow x \in \text{Int}(A^c \cap B^c)$.

Therefore, $\bar{A}^c \cap \bar{B}^c \subseteq \text{Int}(A^c \cap B^c)$.

Thus, $\text{Int}(X \setminus (A \cup B)) = X \setminus (\bar{A} \cup \bar{B})$. □

continued...

② Define the equivalence relation on $\mathbb{R}$ such that $x \sim y$ if $x-y$ is rational. Let $\mathbb{R}/\sim$ be the quotient space with the quotient topology. Show that $\mathbb{R}/\sim$ is not Hausdorff.

Pf: We claim that $\mathbb{R}/\sim$ has the trivial topology.

Let $U \subseteq \mathbb{R}/\sim$ be any nonempty open set. It suffices to show that $U = \mathbb{R}/\sim$.

Let $q: \mathbb{R} \to \mathbb{R}/\sim$ denote the quotient map.

Since q is cts and U is open in $\mathbb{R}/\sim$, we have that $q^{-1}(U)$ is open in $\mathbb{R}$.

Since $q^{-1}(U) \subset \mathbb{R}$ is open and nonempty it contains an open interval, say $(a-\varepsilon, a+\varepsilon)$ for some $a \in \mathbb{R}$ and some $\varepsilon > 0$. It is also closed under the equivalence relation.

Claim: $q^{-1}(U) = \mathbb{R}$.

To show this, let $x \in \mathbb{R}$ be arbitrary.

By density of $\mathbb{Q}$, there exists a rational r with $|x-a-r| < \varepsilon$.

Hence, $x - r = a + x - a - r \in (a-\varepsilon, a+\varepsilon)$.

Since $x - r \sim x$, we conclude that $x \in q^{-1}(U)$.

Since x was arbitrary, we have $q^{-1}(U) = \mathbb{R}$.

Taking the image of both sides yields, $q(q^{-1}(U)) = q(\mathbb{R})$.

Since q is surjective, we see that $U = \mathbb{R}/\sim$.

Therefore, $\mathbb{R}/\sim$ is not Hausdorff.

$\square$

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Let A be an open subset in $\mathbb{R}^n$, $n \geq 2$, whose boundary ∂A is connected. Is ∂A necessarily path-connected? Prove your assertion.

Pf. The boundary ∂A need not be path-connected.

Consider the set $T \subseteq \mathbb{R}^2$ defined by

$$T = \{(x, \sin(\tfrac{1}{x})) : x \in (0,\infty)\} \cup (\{0\} \times [-1,1]).$$

This is the (closure of) the topologist's sine curve (including the segment of the y-axis ensures the set is closed).

We consider the set $A = \mathbb{R}^2 \setminus T$, which is open.

We claim that $\partial A = T$.

- If $x = (x_0, x_1) \in T$ has $x_0 \neq 0$, then it is clear that any ball $B(x,r)$ contains a vertical line segment through x_0. We note that the only vertical line segments contained in T lie on the y-axis. Therefore, x is a boundary point of A.

- If $x = (0, x_1) \in T$, then any open ball $B(x,r)$ contains a point whose first coordinate is negative. Since T only contains points with nonnegative first coordinate, we conclude $B(x,r)$ intersects A, so that $x \in \partial A$.

This establishes $T \subseteq \partial A$.

The equality $T = \partial A$ follows because A is open and hence disjoint from its boundary.

$\square$

continued...

④ Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is null-homotopic, that is, it is homotopic to a constant map.

Pf: Let $f: X \to S^1$ continuous.

We will use the general lifting lemma to show that $\exists$ a lift $\tilde{f}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Observe that X is path-connected and locally path-conn.

Let $p: \mathbb{R} \to S^1$ (the exp. map) be a covering map.

To use the GLL, it suffices to show that $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$.

Observe that $\pi_1(\mathbb{R}) = 0$, so we WTS that $f_*(\pi_1(X)) = 0$.

Since $\pi_1(X)$ is finite, we have that $f_*(\pi_1(X))$ must be finite.

Observe that $f_*(\pi_1(X)) \subseteq \pi_1(S^1) = \mathbb{Z}$. The only finite subgp of $\mathbb{Z}$ is 0.

So $f_*(\pi_1(X)) = 0 \subseteq p_*(\pi_1(\mathbb{R}))$ ✓.

Therefore, by the GLL, there exists a lift $\tilde{f}: X \to \mathbb{R}$.

Recall that a continuous function into a contractible space is null-homotopic.

Since $\tilde{f}$ is cts and $\mathbb{R}$ is contractible, we have that $\tilde{f}$ is null-homotopic.

Since $\tilde{f}$ is null-homotopic, we have that f is null-homotopic.

(If H is the homotopy btwn $\tilde{f}$ and a constant, then $p \circ H$ is the homotopy btwn f and a constant map.)

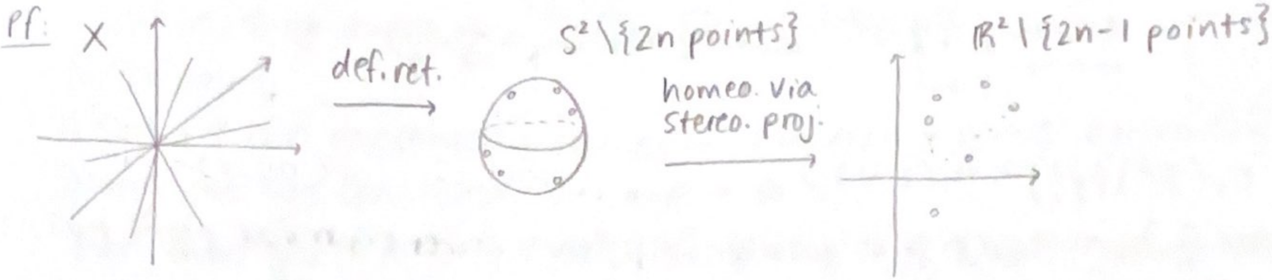
Therefore, f is null-homotopic, i.e., every $f: X \to S^1$ is homotopic to a constant map.

$\square$

...ued.

Compute the fundamental groups of the following spaces:

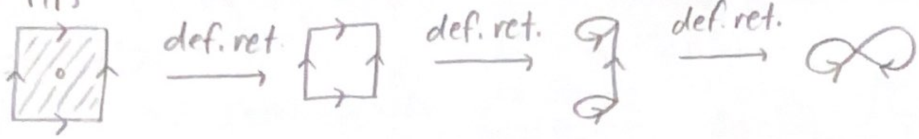
(a) $X \subset \mathbb{R}^3$ is the complement of the union of n lines through the origin.

Pf: 

Continued...

(b) $\mathbb{T}^2 \setminus \{p\}$, where $\mathbb{T}^2$ is the torus.

Pf: $\mathbb{T}^2 \setminus \{p\}$



$= S^1 \vee S^1$

So $\pi_1(\mathbb{T}^2 \setminus \{p\}) = \pi_1(S^1 \vee S^1)$.

Each space, S^1 and S^1, is locally Euclidean, so the wedge point has a nbhd in each space that def. ret. to the wedge point.

Therefore, we can use Van-Kampen as follows: $\pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1)$
$= \mathbb{Z} * \mathbb{Z}$.

Thus, $\pi_1(\mathbb{T}^2 \setminus \{p\}) = \mathbb{Z} * \mathbb{Z}$.

$\square$

...ued...

Prove that every continuous map $h: D \to D$ has a fixed point, that is, a point x with $h(x) = x$. Here D is the closed unit disk in $\mathbb{R}^2$.

Pf: Assume that there is a continuous map $h: D \to D$ that does not have a fixed point.

Then, we can construct a continuous function $g: D \to S^1$ by sending x along the ray connected $h(x)$ and x to S^1.

But then $g(x) = x$ for all $x \in S^1$, so that g is a retraction of D onto S^1.

But no such retraction exists since if one did, then for γ is a loop in S^1.

Since D is convex, there is a homotopy $H: [0,1] \times D \to D$ to a constant loop x_0 where γ is based at x_0.

Since g is the identity on S^1, so $g \circ H$ is a homotopy from $g \circ \gamma = \gamma$ to the constant loop x_0.

So each loop in S^1 is homotopic to a constant loop. ↯

This is a contradiction to $\pi_1(S^1) = \mathbb{Z}$.

Therefore, every continuous map $h: D \to D$ has a fixed point.

□