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Prove or disprove the following statements.

(1) Let A, B be two open subsets in a topological space X. Suppose that $A \cup B$ and $A \cap B$ are connected. Then A and B must be connected.

Pf: We will show that if A or B is disconnected, then $A \cup B$ or $A \cap B$ is disconnected.

WLOG, suppose A is disconnected. Then we can write $A = A_1 \cup A_2$ s.t. A_1, A_2 are open, nonempty, disjoint subsets of A (and also open in X because A is open in X so a subspace of A is open iff it is open in X).

Then we have $A \cup B = (A_1 \cup A_2) \cup B$
$A \cap B = (A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$.

- If both $A_1 \cap B$ and $A_2 \cap B$ are both nonempty, then $A \cap B$ equals the union of two open, nonempty, disjoint sets ($A_1 \cap B \subseteq A_1$ and $A_2 \cap B \subseteq A_2 \Rightarrow A_1 \cap A_2 = \emptyset$), so $A \cap B$ is disconnected.
- If at least one of $A_1 \cap B$ or $A_2 \cap B$ is empty, WLOG suppose $A_1 \cap B = \emptyset$, then we can write $A \cup B = A_1 \cup (A_2 \cup B)$ since $A_1 \cap B = \emptyset$ and $A_1 \cap A_2 = \emptyset$. And both A_1 and A_2 are nonempty and open, so $A \cup B$ is disconnected.

Therefore, we are done. □

(2) Let $\{A_i\}$ be a countable collection of open subsets of a topological space X. Suppose that $\bigcup_i A_i$ and $\bigcap_i A_i$ are connected. Then A_i must be connected for each i.

Pf: False.

Let $X = \mathbb{R}$. Observe that the only connected subsets of $\mathbb{R}$ are intervals, singletons, and $\emptyset$.

Let $A_1 = (0,1) \cup (3,4)$ and $A_i = (0,i)$ for $i \geq 2$.

A_1 and A_i are all open subsets of $\mathbb{R}$.

We have that $\bigcup_{i=1}^{\infty} A_i = (0, \infty)$, which is connected, and $\bigcap_{i=1}^{\infty} A_i = (0,1)$, which is also connected.

The A_i for $i \geq 2$ are connected, but A_1 is clearly disconnected.

Therefore, the statement is false. □

Continued...

② Let $n \geq 2$. Define a topology $\mathcal{Z}$ on $\mathbb{R}^n$ such that every nonempty open set of $\mathcal{Z}$ is the form $\mathbb{R}^n \setminus \{\text{at most finitely many points}\}$. Show that any continuous function $f: (\mathbb{R}^n, \mathcal{Z}) \to \mathbb{R}$ is constant.

Pf: Let Y be a Hausdorff space.

We will show that $f: (\mathbb{R}^n, \mathcal{Z}) \to Y$ continuous is constant.

Assume that f is continuous and nonconstant.

Since f is nonconstant, we know there exist distinct $a, b \in f(\mathbb{R}^n, \mathcal{Z}) \subseteq Y$.

Since Y is Hausdorff and $a \neq b$, we have that $\exists$ open nbhds U of a and V of b such that $U \cap V = \emptyset$.

Since f is continuous and $U, V \subseteq Y$ are open, we know that $f^{-1}(U)$ and $f^{-1}(V)$ are open in $(\mathbb{R}^n, \mathcal{Z})$.

Since $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty open sets in $(\mathbb{R}^n, \mathcal{Z})$, they must be of the form $\mathbb{R}^n \setminus \{\text{at most finitely many points}\}$.

Suppose $f^{-1}(U) = \mathbb{R}^n \setminus \{a_1, \dots, a_n\}$ and $f^{-1}(V) = \mathbb{R}^n \setminus \{b_1, \dots, b_m\}$.

Then $f^{-1}(U) \cap f^{-1}(V) = \mathbb{R}^n \setminus \underbrace{\{a_1, \dots, a_n, b_1, \dots, b_m\}}_{\text{finitely many points}}$

$\neq \emptyset$ b/c $\mathbb{R}^n$ is infinite

But $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. ↯

This is a contradiction.

Therefore, f must be constant.

Since $\mathbb{R}$ is Hausdorff, we have shown that any continuous function $f: (\mathbb{R}^n, \mathcal{Z}) \to \mathbb{R}$ is constant.

□

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Let $f: X \to Y$ be a continuous and injective map between topological spaces X and Y. Prove that if X is compact and Y is Hausdorff, then f is an embedding.

Pf: Recall that f is an embedding if the restriction $\tilde{f}: X \to f(X)$ is a homeo.

Since f is continuous and injective, it suffices to show that f is closed.

Let $K \subseteq X$ be a closed subset.

Closed subsets of compact spaces are compact.

Since K is closed, X is compact, and $K \subseteq X$, we have that K is compact in X.

The continuous image of a compact set is compact.

Since f is continuous and K is compact, we have that $f(K)$ is compact in Y.

Compact subsets of Hausdorff spaces are closed.

Since $f(K)$ is compact, Y is Hausdorff, and $f(K) \subseteq Y$, we have that $f(K)$ is closed in Y.

Therefore, if $K \subseteq X$ is closed, then $f(K) \subseteq Y$ is closed $\Rightarrow$ f is a closed map.

Thus, if X is compact and Y is Hausdorff, then f is an embedding.

$\square$

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④ Define $M = \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 - 1\}$ with the induced topology from $\mathbb{R}^3$.

(1) Find a universal covering of M.

Pf: Observe that M is homeomorphic to a cylinder $S^1 \times \mathbb{R}$ by the map $f: M \to S^1 \times \mathbb{R}$ given by $f(x,y,z) = \left(\frac{x}{\sqrt{z^2+1}}, \frac{y}{\sqrt{z^2+1}}, z\right)$.

It is clear that f is continuous. We also have $f^{-1}: S^1 \times \mathbb{R} \to M$ given by $f^{-1}(x,y,z) = (x\sqrt{z^2+1}, y\sqrt{z^2+1}, z)$, which is also continuous.

First we will check that f actually carries M into $S^1 \times \mathbb{R}$:

if $(x,y,z) \in M$, then $f(x,y,z) = \left(\frac{x}{\sqrt{z^2+1}}, \frac{y}{\sqrt{z^2+1}}, z\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right) \in S^1 \times \mathbb{R}$.

Now we will check that f^{-1} actually carries $S^1 \times \mathbb{R}$ into M:

if $(x,y,z) \in S^1 \times \mathbb{R}$, then $f^{-1}(x,y,z) = (x\sqrt{z^2+1}, y\sqrt{z^2+1}, z)$, so that the sum of the squares of the first two coordinates is $(z^2+1)(x^2+y^2) = z^2+1$. ✓

Since homeomorphic spaces have the same universal covering space, it suffices to determine the covering space of the cylinder.

Products of covering maps are covering maps, so it suffices to find the universal cover of S^1 and $\mathbb{R}$, both of which are $\mathbb{R}$.

So $\mathbb{R}^2$ is the universal covering space of the cylinder (because $\mathbb{R}^2$ is simply connected). □

(2) Let $X = M/\sim$ be the quotient space where $\sim$ is the equivalence relation generated by the relation $(x,y,z) \sim (x,y,-z)$. Is the quotient map $q: M \to X$ a covering map? Explain your answer.

Pf: No, q is not a covering map.

Observe that M is connected (it is homeo. to a cylinder, which is connected).

If a covering map has a connected domain, then every fiber of the map has the same cardinality.

Therefore, if q is covering, each fiber $q^{-1}(x)$ must have the same size.

But $q^{-1}([1,1,1]) = \{(1,1,1), (1,1,-1)\}$ has size 2, whereas

$q^{-1}([1,0,0]) = \{(1,0,0)\}$ has size 1. ↯

This is a contradiction.

Thus, q cannot be a covering map. □

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Let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and let D denote the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Let X be a topological space. Prove that the following statements are equivalent:

(1) For every point $x \in X$, the fundamental group $\pi_1(X,x)$ is trivial.

(2) For every continuous function $f: S^1 \to X$, there exists a continuous map $F: D \to X$ which extends f.

Pf: (2) ⇒ (1): Let $x \in X$ and $[f] \in \pi_1(X,x)$.

Since f is a loop, let ω be its circle representative, i.e., the map that it descends to out of the quotient.

By assumption, ω extends to a cts $F: S^1 \to D$.

For each $s \in [0,1]$, let $H_s(t)$ be the straight-line path in D from 1 to $e^{2\pi i s}$. Then set $H(s,t) = H_s(t) = (1-t)\cdot 1 + t \cdot e^{2\pi i s}$.

We claim that $F \circ H$ is a path homotopy between $[f]$ and the constant path at $f(0) = x$. $F \circ H$ is clearly cts. $F \circ H$ is a homotopy:

$F \circ H(s,0) = F(1) = \omega(1) = f(0)$

$F \circ H(s,1) = F(e^{2\pi i s}) = \omega(e^{2\pi i s}) = f(s)$

$F \circ H(0,t) = F(1-t+t) = F(1) = \omega(1) = f(0)$

$F \circ H(1,t) = F(1-t+t) = F(1) = \omega(1) = f(0)$

Therefore, $[f]$ is the class of the trivial loop, so $\pi_1(X,x)$ is trivial.

(1) ⇒ (2): Let $f: S^1 \to X$ be a continuous map.

Then $f \circ q: [0,1] \to X$ is a loop in X, where $q: [0,1] \to S^1$ is the usual quotient map $q(s) = e^{2\pi i s}$.

By assumption, there is a path homotopy $H: [0,1] \times [0,1] \to X$ such that $H_1(s) = f \circ q(s)$ and $H_0(s) = f(1)$.

We see that, by defn. of path homotopy, $H(0,t) = f(1) = H(1,t)$.

Claim: $F(te^{2\pi i s}) = H(s,t)$ is an extension of f. (Well-def b/c $H(0,t) = H(1,t)$)

It is clearly cts.

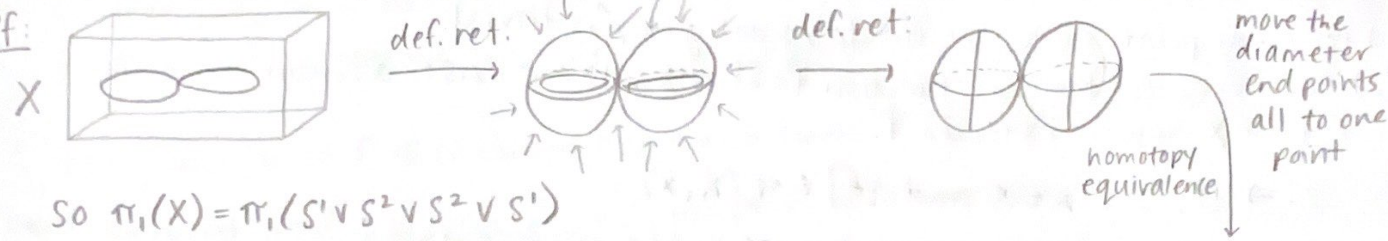
We see that $F(e^{2\pi i s}) = H_1(s) = f \circ q(s) = f(e^{2\pi i s})$, so this really is an extension of f.

□

Continued...

⑥ Let X be the space obtained from $\mathbb{R}^3$ by removing two circles $C_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+(y-1)^2 = 1,\ z=0\}$ and $C_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+(y+1)^2 = 1,\ z=0\}$. Compute $\pi_1(X)$ and justify your answer.

Pf:



So $\pi_1(X) = \pi_1(S^1 \vee S^2 \vee S^2 \vee S^1)$

Each space, S^1 and S^2, is locally Euclidean, so the wedge point has a nbhd in each space that deformation retracts to the wedge point (itself).

$S^1 \vee S^2 \vee S^2 \vee S^1 =$ 

Therefore, we can use the following version of Van-Kampen:

$$\begin{aligned}\pi_1(S^1 \vee S^2 \vee S^2 \vee S^1) &= \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^2) * \pi_1(S^1)\\ &= \mathbb{Z} * 0 * 0 * \mathbb{Z}\\ &= \mathbb{Z} * \mathbb{Z}\end{aligned}$$

Thus, $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$. □