ust 2021

(1) A topological space X is said to be metrizable if its topology is generated by some metric on X. Does every compact metrizable space X have a countable basis? Prove your assertion.

Pf: Recall that if X is a metrizable space, then
X second countable $\Leftrightarrow$ X separable $\Leftrightarrow$ X Lindëloff.

Let X be a compact metrizable space.

Compact spaces are always Lindëloff.

Since X is compact, X is Lindëloff.

In a metrizable space X, X is second countable $\Leftrightarrow$ X is Lindëoff.

Since X is metrizable and Lindëloff, we have that X is second countable.

Therefore, every compact metrizable space X has a countable basis (i.e., is second countable). $\square$

continued...

(2) Suppose $q: X \to Y$ is an open quotient map. Prove that Y is Hausdorff if and only if the set $R = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

Pf: • Suppose Y is Hausdorff.

We WTS that $R = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

We will do this by showing that $X \times X \setminus R = \{(x_1, x_2) : q(x_1) \neq q(x_2)\}$ is open.

Let $(x_1, x_2) \in X \times X \setminus R$.

Then $q(x_1), q(x_2) \in Y$ s.t. $q(x_1) \neq q(x_2)$.

Since Y is Hausdorff and $q(x_1) \neq q(x_2)$, $\exists$ open nbhds U of $q(x_1)$ and V of $q(x_2)$ s.t. $U \cap V = \emptyset$.

Since q is cts and $U, V \subseteq Y$ are open, we have that $q^{-1}(U)$ and $q^{-1}(V)$ are open in X.

Observe that $x_1 \in q^{-1}(U)$ and $x_2 \in q^{-1}(V)$.

Let $W := q^{-1}(U) \times q^{-1}(V)$.

Clearly $(x_1, x_2) \in q^{-1}(U) \times q^{-1}(V) = W$.

W is open since it is the product of two open sets.

We WTS that $W \cap R = \emptyset$:

Let $(y_1, y_2) \in W = q^{-1}(U) \times q^{-1}(V)$.

$$\left.\begin{array}{l}\text{Since } y_1 \in q^{-1}(U), \text{ we have } q(y_1) \in U \\ \text{Since } y_2 \in q^{-1}(V), \text{ we have } q(y_2) \in V\end{array}\right\} U \cap V = \emptyset, \text{ so } q(y_1) \neq q(y_2) \text{ for all } (y_1, y_2) \in W.$$

Therefore, $W \cap R = \emptyset$.

So we have $(x_1, x_2) \in \overset{\text{open}}{W} \subseteq X \times X \setminus R$.

Thus, $X \times X \setminus R$ is open $\Rightarrow R$ is closed.

We conclude that if Y is Hausdorff, then the set $R = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

• Suppose $R = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

We WTS that Y is Hausdorff: for $y_1, y_2 \in Y$ $(y_1 \neq y_2)$, $\exists$ open nbhds U of y_1 and V of y_2 s.t. $U \cap V = \emptyset$.

Let $y_1, y_2 \in Y$ s.t. $y_1 \neq y_2$.

Since q is surjective, we know $\exists x_1, x_2 \in X$ s.t. $q(x_1) = y_1$ and $q(x_2) = y_2$.

$(q(x_1) \neq q(x_2)$ since $y_1 \neq y_2)$.

...ued...

So we know that $(x_1, x_2) \notin R$, i.e., $(x_1, x_2) \in X \times X \setminus R$ which is open since R is closed.

Since $X \times X \setminus R$ is open, we have that $(x_1, x_2) \in B \subseteq X \times X \setminus R$, where B is a basis element for the product topology.

We know that a basis for the product topology is $\{U \times V : U, V \text{ are open in } X\}$, so $B = U \times V$ where U, V are open in X.

Since $(x_1, x_2) \in B$, we have that $x_1 \in U$, $x_2 \in V$ where U, V are open in X from B.

Since q is open, we have that $q(U)$ and $q(V)$ are open in Y.

Observe that $\underset{\overset{\shortparallel}{y_1}}{q(x_1)} \in q(U)$ and $\underset{\overset{\shortparallel}{y_2}}{q(x_2)} \in q(V)$

So $y_1 \in q(U)$ and $y_2 \in q(V)$, i.e., $q(U)$ is an open nbhd of y_1 and
$q(V)$ is an open nbhd of y_2.

We WTS that $q(U) \cap q(V) = \emptyset$.

Recall that U, V are open sets, $B = U \times V$ and $B \subseteq X \times X \setminus R$, so if $(x_1, x_2) \in B$, then $q(x_1) \neq q(x_2)$.

Let $y \in q(U) \cap q(V)$.

$$\left.\begin{array}{l} \text{Since } y \in q(U),\ \exists\, x_1 \in U \text{ s.t. } q(x_1) = y \\ \text{Since } y \in q(V),\ \exists\, x_2 \in V \text{ s.t. } q(x_2) = y \end{array}\right\} \underbrace{q(x_1) = q(x_2) = y}_{\text{This cannot happen by definition of } B.} \ \lightning$$

Therefore, $q(U) \cap q(V) = \emptyset$.

Thus, for $y_1, y_2 \in Y$ $(y_1 \neq y_2)$, $\exists$ open nbhds $q(U)$ and $q(V)$ s.t. $q(U) \cap q(V) = \emptyset$.

We conclude that if $R = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$, then Y is Hausdorff.

$\square$

continued...

③ Prove that a space X is contractible if and only if every map $f: X \to Y$, for an arbitrary Y, is null-homotopic.

Pf: • Suppose $f: X \to Y$, for an arbitrary Y, is null-homotopic.
Let $Y = X$, so $f: X \to X$ is $id_X: X \to X$, which is null-homotopic.
Therefore, X is contractible.

• Suppose X is contractible.
We WTS that $f: X \to Y$ is null-homotopic.
Since X is contractible, we have that $id_X: X \to X$ is homotopic to a constant map, c_{x_0}.
Let $H: [0,1] \times X \to X$ be the homotopy, so $H(0,x) = id_X(x) = x$,
$H(1,x) = c_{x_0}(x) = x_0$.

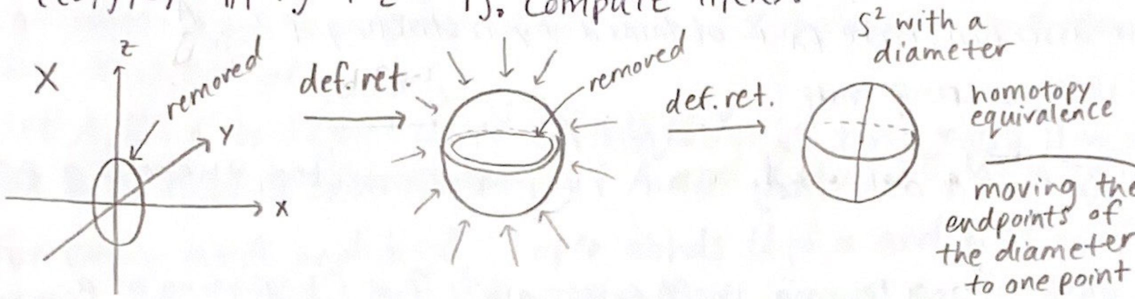
Let $\tilde{H}: [0,1] \times X \to Y$ be defined by $\tilde{H}(t,x) = (f \circ H)(t,x)$.
Observe that $\tilde{H}$ is continuous since it's the composition of continuous functions.
Observe that $\tilde{H}(0,x) = f(H(0,x)) = f(x)$
and $\tilde{H}(1,x) = f(H(1,x)) = f(x_0)$ fixed.
Therefore, $\tilde{H}$ is a homotopy between f and a constant map.
Thus, every map $f: X \to Y$, for an arbitrary Y, is null-homotopic.

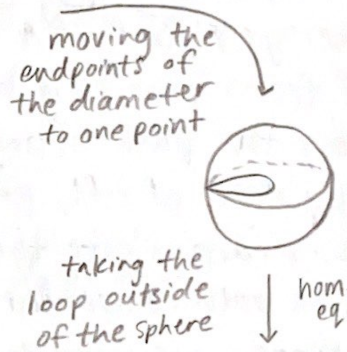
□

...nued...

① Let X be the space obtained from $\mathbb{R}^3$ by removing the circle $C = \{(0,y,z) \in \mathbb{R}^3 : y^2 + z^2 = 4\}$. Compute $\pi_1(X)$.

Pf:





Therefore, we have $\pi_1(X) = \pi_1(S^1 \vee S^2)$

$S^1 \vee S^2 =$ 

Let $U =$  $\xrightarrow{\text{def. ret.}}$  $= S^2$

U is open, path-connected. $\pi_1(U) = \pi_1(S^2) = 0$

Let $V =$  $\xrightarrow{\text{def. ret.}}$  $= S^1$

V is open, path-connected. $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$

Observe that $S^1 \vee S^2 = U \cup V$.

$U \cap V =$  $\xrightarrow{\text{def. ret.}} \cdot$

$U \cap V$ is path-connected, nonempty. $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(S^1 \vee S^2) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = 0 * \mathbb{Z} = \mathbb{Z}$.

Therefore, $\pi_1(X) = \pi_1(S^1 \vee S^2) = \mathbb{Z}$.

Thus, $\pi_1(X) = \mathbb{Z}$. $\square$

Continued...

(5) Let $p: X \to Y$ be a covering map and Y be path-connected and locally path connected. If $A \subseteq X$ is a path component of X, is $p_A : A \to Y$ a covering map? Prove your assertion. Here p_A is obtained by restricting p on A.

Pf: Yes, p_A is a covering map.

First, we will prove that p_A is surjective.

Let $y \in Y$ and $a \in A$ arbitrary. Since Y is path-connected, there is a path f from $p(a)$ to y.

By the path-lifting lemma, there exists a lift $\tilde{f}$ of f beginning at a.

By defn. of lift, $p \circ \tilde{f}(1) = f(1) = y$.

So p carries out the point $\tilde{f}(1)$ to y. Since $\tilde{f}$ is a path in X that intersects the path component A, $\tilde{f}([0,1]) \subset A$, so $\tilde{f}(1) \in A$.

· Finally, we need to show that each $y \in Y$ has a p_A-evenly covered nbhd.

Let $U \subseteq Y$ be any p-evenly covered nbhd of Y.

Since Y is locally path-connected, there exists a nbhd W of y such that $W \subset U$ and W is path-connected.

Recalling that an open subset of a p-evenly covered nbhd is p-evenly covered, we conclude that W is p-evenly covered.

· Finally, we will show that W is p_A-evenly covered as well.

We know that $p^{-1}(W) = \bigcup_{\alpha \in I} V_\alpha$ where the V_α are open, pairwise disjoint, and $p: V_\alpha \to W$ is a homeomorphism and I is some index set.

In particular, this means each V_α is path-connected because W is path-connected. Notice that $p_A^{-1}(W) = p^{-1}(W) \cap A = \bigcup_{\alpha \in I} V_\alpha \cap A$.

Since A is a path-component and V_α is path-connected, we see either $V_\alpha \cap A = V_\alpha$ or $V_\alpha \cap A = \emptyset$.

If we define $J = \{\alpha \in I : V_\alpha \cap A = V_\alpha\}$, then $p_A^{-1}(W) = \bigcup_{\alpha \in J} V_\alpha$.

Therefore, p_A is a covering map.

$\square$

...ued...

A topological space X is said to be normal if it is Hausdorff and for every pair of disjoint closed subsets $A, B \subseteq X$, there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. Prove that if X is compact and Hausdorff, then X is normal.

Pf: Let $A, B \subseteq X$ be disjoint closed subsets of X.

We WTS that $\exists$ open nbhds U of A and V of B s.t. $U \cap V = \emptyset$.

- For every $a \in A$ and $b \in B$, $\exists$ open nbhds U of a and V of b s.t. $U \cap V = \emptyset$ because X is Hausdorff.
- Fix $x \in X$ and $A \subseteq X$ closed s.t. $x \notin A$.

  For each $a \in A$, we have open nbhds U_a of x and V_a of a s.t. $U_a \cap V_a = \emptyset$.

  Observe that A is compact since it is a closed subset of a compact space.

  Since A is compact, we have that for a given open cover $\{V_a : a \in A\}$, there is a finite subcover $\{V_{a_i} : a_i \in A, 1 \le i \le n\}$.

  Let $V = \bigcup_{i=1}^{n} V_{a_i}$ and $U = \bigcap_{i=1}^{n} U_{a_i}$.

  Observe that $V \neq \emptyset$ b/c $A \subseteq V$ and $U \neq \emptyset$ b/c $x \in U$.

  We also have that V is open b/c the union of arbitrarily many open sets is open, and U is open b/c the finite intersection of open sets is open.

  So we have that U is an open nbhd of $x \in X$ and V is an open nbhd of $A \subseteq X$ s.t. $U \cap V = \emptyset$ b/c $U_a \cap V_a = \emptyset \; \forall a \in A$.

  Therefore, X is regular.
- For each $a \in A$, we have open nbhds U_a of a and V_a of B s.t. $U_a \cap V_a = \emptyset$ because X is regular.

  Observe that A is cpt since it is a closed subset of a cpt space.

  Since A is cpt, we have that for a given open cover $\{U_a : a \in A\}$, there is a finite subcover $\{U_{a_i} : a_i \in A, 1 \le i \le n\}$.

  Let $U = \bigcup_{i=1}^{n} U_{a_i}$ and $V = \bigcap_{i=1}^{n} V_{a_i}$.

  Observe that $U \neq \emptyset$ b/c $A \subseteq U$ and $V \neq \emptyset$ b/c $b \in V$ where $b \in B$. $(B \subseteq V)$

  We also have that U is open (union of arb. open sets) and V is open (finite intersection of open sets).

  So U is an open nbhd of A and V is an open nbhd of B s.t. $U \cap V = \emptyset$.

  b/c $U_a \cap V_a = \emptyset \; \forall a \in A$.

  Therefore, X is normal.

$\square$