Topology HW 1

① Let A and B denote subsets of a topological space X. Prove that

$$\overline{A \cup B} = \bar{A} \cup \bar{B}.$$

Pf. First we will show that $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Observe that $A \subseteq A \cup B \subseteq \overline{A \cup B}$, so $\overline{A \cup B}$ is a closed set containing A. Note that $\bar{A}$ is the smallest closed set containing A, so we have that $\bar{A} \subseteq \overline{A \cup B}$.

Likewise, observe that $B \subseteq A \cup B \subseteq \overline{A \cup B}$, so $\overline{A \cup B}$ is a closed set containing B. Observe that $\bar{B}$ is the smallest closed set containing B, so we have that $\bar{B} \subseteq \overline{A \cup B}$.

Therefore, $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Now we will show that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

We will show this by proving that if $x \notin \bar{A} \cup \bar{B}$, then $x \notin \overline{A \cup B}$.

Suppose $x \notin \bar{A} \cup \bar{B}$.

If $x \notin \bar{A}$, then $\exists$ an open nbhd U of x s.t. $U \cap A = \emptyset$.

If $x \notin \bar{B}$, then $\exists$ an open nbhd V of x s.t. $V \cap B = \emptyset$.

Observe that $U \cap V$ is open since the finite intersection of open sets is open. We also have that $U \cap V \neq \emptyset$ since $x \in U \cap V$.

So $U \cap V$ is an open nbhd of x.

Observe that $(U \cap V) \cap (A \cup B) = \emptyset$ since $U \cap V \subseteq U$ and $U \cap A = \emptyset$
$U \cap V \subseteq V$ and $V \cap B = \emptyset$.

Therefore, $x \notin \overline{A \cup B}$ because we have an open nbhd of x that is disjoint from $A \cup B$.

Thus, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

We conclude that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

□

continued...

② Let $\mathbb{R}_\ell$ be the set of real numbers $\mathbb{R}$ associated with the topology given by the basis $\mathcal{B} = \{[a,b);\ a<b,\ a,b \in \mathbb{Q}\}$. Determine the closure of the subsets $(1,\sqrt{2})$ and $(\sqrt{2},3)$ in $\mathbb{R}_\ell$.

Pf: • First we will determine the closure of $(1,\sqrt{2})$ in $\mathbb{R}_\ell$.

Observe that $(1,\sqrt{2}) \subseteq \overline{(1,\sqrt{2})}$.

- If $x<1$, then an open nbhd of x is $[a,1)$, where $a \in \mathbb{Q}$, $a \leq x$.
  Notice that $[a,1) \cap (1,\sqrt{2}) = \emptyset$.
  Therefore, every $x<1$ is not in $\overline{(1,\sqrt{2})}$.
- If $x>\sqrt{2}$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $\sqrt{2}<a<x<b$. Notice that $(1,\sqrt{2}) \cap [a,b) = \emptyset$.
  Therefore, every $x>\sqrt{2}$ is not in $\overline{(1,\sqrt{2})}$.
- If $x=1$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $a \leq \overset{1}{x} < b$.
  Notice that $[a,b) \cap (1,\sqrt{2}) = (1,c) \neq \emptyset$, where $c = \min\{b,\sqrt{2}\}$.
  Therefore, $x=1$ is in $\overline{(1,\sqrt{2})}$.
- If $x=\sqrt{2}$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $a < \overset{\sqrt{2}}{x} < b$.
  Notice that $(1,\sqrt{2}) \cap [a,b) = (c,\sqrt{2}) \neq \emptyset$, where $c = \max\{1,a\}$.
  Therefore, $x=\sqrt{2}$ is in $\overline{(1,\sqrt{2})}$.

Thus, we conclude that $\overline{(1,\sqrt{2})} = [1,\sqrt{2}]$ in $\mathbb{R}_\ell$.

• Now we will determine the closure of $(\sqrt{2},3)$ in $\mathbb{R}_\ell$.

Observe that $(\sqrt{2},3) \subseteq \overline{(\sqrt{2},3)}$.

- If $x<\sqrt{2}$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $a \leq x < b < \sqrt{2}$.
  Notice that $[a,b) \cap (\sqrt{2},3) = \emptyset$.
  Therefore, every $x<\sqrt{2}$ is not in $\overline{(\sqrt{2},3)}$.
- If $x>3$, then an open nbhd of x is $[3,b)$, where $b \in \mathbb{Q}$, $3<x<b$.
  Notice that $(\sqrt{2},3) \cap [3,b) = \emptyset$.
  Therefore, every $x>3$ is not in $\overline{(\sqrt{2},3)}$.
- If $x=\sqrt{2}$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $a < \overset{\sqrt{2}}{x} < b$.
  Notice that $[a,b) \cap (\sqrt{2},3) = (\sqrt{2},c) \neq \emptyset$, where $c = \min\{b,3\}$.
  Therefore, $x=\sqrt{2}$ is in $\overline{(\sqrt{2},3)}$.
- If $x=3$, then an open nbhd of x is $[a,b)$, where $a,b \in \mathbb{Q}$, $a \leq 3 < b$.
  Notice that $(\sqrt{2},3) \cap [a,b) \neq \emptyset$ if $\sqrt{2}<a<3$.
  Therefore, $x=3$ is not in $\overline{(\sqrt{2},3)}$.

Thus, we conclude that $\overline{(\sqrt{2},3)} = [\sqrt{2},3)$ in $\mathbb{R}_\ell$. □

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Recall that a space X is first countable if for each $x \in X$, there is a countable collection $\mathcal{O}_x$ of open sets containing x such that whenever U is an open set containing x, there exists some $V \in \mathcal{O}_x$ such that $V \subset U$.
Let X be a first countable topological space and $x \in X$ and $A \subset X$. Prove that $x \in \bar{A}$ if and only if there exists a sequence of points in A converging to x in X.

Pf: • Suppose there exists a sequence of points in A converging to x in X.
Let $\{x_n\}_{n=1}^{\infty} \in A$ s.t. $x_n \to x \in X$.
We WTS that $x \in \bar{A}$, i.e., that every nbhd of x intersects A.
Let U be an open nbhd of x. Then there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$ we have $x_n \in U$. Therefore, $U \cap A \neq \emptyset$ (because $x_n \in A$ for $n \geq N$).
Therefore, $x \in \bar{A}$.

• Suppose that $x \in \bar{A}$.
Let $\mathcal{O}_x = \{V_n\}_{n=1}^{\infty}$ be a local nbhd basis at x.
We want to construct a sequence in A that converges to x in X.
Observe that $V_1 \cap A \neq \emptyset, \dots, V_n \cap A \neq \emptyset$ for all $n \in \mathbb{N}$ since $x \in \bar{A}$.
Let $x_1 \in V_1 \cap A \neq \emptyset$, $x_2 \in V_2 \cap A \neq \emptyset, \dots, x_n \in V_n \cap A \neq \emptyset$.
We know that each $\bigcap_{i=1}^{n} V_i \cap A \neq \emptyset$ because each $\bigcap_{i=1}^{n} V_i$ is a nbhd of x, so it must intersect A. Therefore, we have a sequence $\{x_n\}_{n=1}^{\infty} \in A$.
It remains to show that $x_n \to x$.
Let U be an open nbhd of x.
Then since X is first countable, we know there exists some $V_N \in \{V_n\}_{n=1}^{\infty}$ for some $N \in \mathbb{N}$ s.t. $V_N \subset U$.
If $n \geq N$, then $x_n \in V_1 \cap \dots \cap V_n \subset V_1 \cap \dots \cap V_N \subset U$.
We have that for all nbhds U of x, there exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies $x_n \in U$.
Therefore, we have constructed a sequence $\{x_n\}_{n=1}^{\infty}$ of points in A that converges to $x \in X$. $\square$

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④ Prove or disprove the following statements.

(i) Let X be a topological space and $C \subset X$ be a closed subset. Then C is equal to the closure of its interior, i.e., $C = \overline{\text{int}\, C}$.

Pf. This statement is false.

Let $X = \mathbb{R}$ and consider the standard topology.

Let $C = \{0\} \subset X$. A single point is a closed subset in X, so C is closed.

Observe that $\text{int}\, C = \text{int}\{0\} = \emptyset$, and $\overline{\text{int}\, C} = \overline{\emptyset} = \emptyset$.

Therefore, we have $C = \{0\} \neq \emptyset = \overline{\text{int}\, C}$. $\square$

(ii) The countable collection $\mathcal{B} = \{(a,b);\ a<b,\ a,b \in \mathbb{Q}\}$ is a basis that generates the standard topology on the real line $\mathbb{R}$.

Pf: First we will prove that $\mathcal{B}$ is a basis: for all $x \in \mathbb{R}$, for all nbhd U of x, there exists $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.

Recall that the basis of the standard topology is composed of all open intervals on the real line $\mathbb{R}$. The elements of the standard topology are just open sets.

- Let $x \in \mathbb{R}$ and let U be an open nbhd of x.

Since U is open, it must contain an open interval, say (a,b), $a,b \in \mathbb{R}$, $a<b$. So $x \in (a,b) \subset U$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know there exists a basis element $(c,d) \in \mathcal{B}$ s.t. $c,d \in \mathbb{Q}$ and $a \leq c < d \leq b$, so $x \in (c,d) \subseteq (a,b) \subset U$.

Since x and U were arbitrarily chosen, we have that $\mathcal{B}$ is indeed a basis for $\mathbb{R}$.

- We have shown that the topology generated by $\mathcal{B}$ is larger than or equal to the standard topology, so it remains to show that $\mathcal{B}$ is contained in the standard topology.

Every open set is contained in the standard topology.

Therefore, $\mathcal{B}$ is contained in the standard topology because $\mathcal{B}$ is composed of open intervals which are open sets. $\square$

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(i) Let X be a set and let $\tau = \{U \subset X ; X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Show that τ is a topology. We call (X, τ) the cofinite topology of X.

Pf: · Observe that $\emptyset \in \tau$ by definition of τ.

· We also have that $X \in \tau$ because $X \subset X$ s.t. $X \setminus X = \emptyset$ is finite.

· We WTS that the union of arbitrarily many open sets is contained in τ. Let $\{V_\alpha\}_{\alpha \in A} \subset \tau$. So $X \setminus V_\alpha$ is finite for every $\alpha \in A$. Then

$$X \setminus \bigcup_{\alpha \in A} V_\alpha = \bigcap_{\alpha \in A} (X \setminus V_\alpha) \subset X \setminus V_\alpha$$ is finite since each $X \setminus V_\alpha$ is finite.

Therefore, $\bigcup_{\alpha \in A} V_\alpha \in \tau$.

· We WTS that the finite intersection of open sets is contained in τ. Let $V_1, \ldots, V_m \in \tau$. So $X \setminus V_i$ is finite for every $1 \leq i \leq m$. Then

$$X \setminus \left(\bigcap_{i=1}^{m} V_i \right) = \bigcup_{i=1}^{m} (X \setminus V_i)$$ is finite since the finite union of finite sets is finite. Therefore, $\bigcap_{i=1}^{m} V_i \in \tau$.

· Thus, we have shown that τ is a topology. $\square$

(ii) Let τ be the cofinite topology on the set $\mathbb{Z}$ of integers. Show that the sequence $\{1, 2, 3, \ldots\}$ of positive integers converges to every point of $\mathbb{Z}$ in $(\mathbb{Z}, \tau)$.

Pf: We WTS that there exists k such that every integer greater than k is in U.

Take $n \in \mathbb{Z}$ arbitrary. Let U be an arbitrary nbhd of n, $U \in \tau$.

Then $X \setminus U$ is finite.

Since $X \setminus U$ is finite, $X \setminus U$ must have a maximum element M s.t. every integer greater than M is in U.

Therefore, the tailend of the sequence is contained in U, i.e., $x_k \to n$.

We can do this for all integers since n was chosen arbitrarily. $\square$

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⑥ A topological space X is called metrizable if the topology is the metric topology associated with some metric on X.

(i) Show that a metrizable topological space X is Hausdorff; that is, for any distinct two points x and y of X, there are open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Pf. Let (X, d) be a metrizable space, where d is the metric.

We WTS that (X, d) is Hausdorff.

Let $x_0, y_0 \in X$ s.t. $x_0 \neq y_0$.

Let U be an open nbhd of x s.t. $U = \{x \in X : d(x, x_0) < \frac{1}{2} d(x_0, y_0)\}$.

Let V be an open nbhd of x s.t. $V = \{y \in X : d(y, y_0) < \frac{1}{2} d(x_0, y_0)\}$.

We WTS that $U \cap V = \emptyset$.

Assume $\exists\, z \in U \cap V$. Then $d(x_0, z) < \frac{1}{2} d(x_0, y_0)$ and $d(y_0, z) < \frac{1}{2} d(x_0, y_0)$.

Observe that $d(x_0, y_0) \leq d(x_0, z) + d(y_0, z) < d(x_0, y_0)$, so we have that $d(x_0, y_0) < d(x_0, y_0)$. $\lightning$ contradiction since $d(x_0, y_0) = d(x_0, y_0)$.

Therefore, $U \cap V = \emptyset$.

Thus, we have shown that for any $x_0, y_0 \in X$ $(x_0 \neq y_0)$, $\exists$ open sets U of x_0 and V of y_0 s.t. $U \cap V = \emptyset$, i.e., (X, d) is Hausdorff. □

(ii) Let X be an infinite set with the cofinite topology τ. Show that (X, τ) is not Hausdorff. Conclude that (X, τ) is not metrizable.

Pf: Suppose that (X, τ) is Hausdorff.

Then for any distinct $x, y \in X$, there exist open nbhds U of x and V of y s.t. $U \cap V = \emptyset$.

Since U, V are open in τ, we have that $X \setminus U, X \setminus V$ are finite.

$\Rightarrow U, V$ are infinite (since X is infinite and $X \setminus U, X \setminus V$ are finite).

Since $U \cap V = \emptyset$, we have that $U \subseteq X \setminus V$ and $V \subseteq X \setminus U$. $\lightning$

Contradiction, because an infinite set cannot be a subset of or equal to a finite set.

Therefore, we conclude that (X, τ) is not Hausdorff. □