Topology HW 2

① Let $f_1, f_2 : X \to Y$ be continuous map from a topological space X to a Hausdorff Space Y.

(i) Show that the set of points $\{x \in X : f_1(x) = f_2(x)\}$ is a closed set.

Pf: Let $A = \{x \in X : f_1(x) = f_2(x)\}$. We WTS that A is closed.

We will do this by showing that $X \setminus A = \{x \in X : f_1(x) \neq f_2(x)\}$ is open.

Let $x \in X \setminus A$. Then $f_1(x), f_2(x) \in Y$ s.t. $f_1(x) \neq f_2(x)$.

Since Y is Hausdorff and $f_1(x) \neq f_2(x)$, we have that there exist open nbhds U of $f_1(x)$ and V of $f_2(x)$ s.t. $U \cap V = \emptyset$.

Since f_1, f_2 are continuous and U, V are open in Y, we have that $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in X.

Observe that $x \in f_1^{-1}(U)$ and $x \in f_2^{-1}(V)$.

Let $W := f_1^{-1}(U) \cap f_2^{-1}(V)$. So $x \in W \neq \emptyset$, i.e., W is nonempty.

W is open since the finite intersection of open sets is open.

We have that W is an open nbhd of x.

We WTS that $W \cap A = \emptyset$.

Let $y \in f_1^{-1}(U) \cap f_2^{-1}(V) = W$.

$$\left.\begin{array}{l} \text{Since } y \in f_1^{-1}(U) \Rightarrow f_1(y) \in U \\ \text{Since } y \in f_2^{-1}(V) \Rightarrow f_2(y) \in V \end{array}\right\} \begin{array}{l} \text{but } U \cap V = \emptyset \\ \text{so } f_1(y) \neq f_2(y) \ \forall y \in W. \end{array}$$

Therefore, $W \cap A = \emptyset$.

Thus, we have $x \in \underset{\text{open}}{W} \subseteq X \setminus A$.

We conclude that $X \setminus A$ is open $\Rightarrow$ A is closed. □

(ii) If there exists a dense subset D of X such that $f_1(x) = f_2(x)$ for all $x \in D$, then $f_1(x) = f_2(x)$ on X.

Pf: Recall that if $D \subseteq X$ is dense, then $\overline{D} = X$.

We WTS that $f_1(x) = f_2(x) \ \forall x \in \overline{D} = X$.

From part (i), we have that $A = \{x \in X : f_1(x) = f_2(x)\}$ is closed.

Observe that A contains a dense subset D, so $D \subseteq A \subseteq X$.

If D is dense in X and A is closed in X, then $A = X$.

Therefore, $f_1(x) = f_2(x) \ \forall x \in X$. □

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② Let X and Y be topological spaces, and Y Hausdorff. Let $A \subset X$ be a nonempty set. Suppose that $f: A \to Y$ is continuous, where A is equipped with the subspace topology. Prove that if there is a continuous extension of f to $\bar{A}$, then the extension is unique.

Pf: A continuous extension of f to $\bar{A}$ is a continuous function g on $\bar{A}$ such that its restriction to A is equal to f, i.e., $g: \bar{A} \to Y$ s.t. $g|_A = f: A \to Y$.

· Assume that the continuous extension of f to $\bar{A}$ is not unique, i.e., there exist continuous extensions $g_1, g_2: \bar{A} \to Y$ of f to $\bar{A}$ such that $g_1|_A = g_2|_A = f$.

Let $x \in \bar{A}$ s.t. $g_1(x) \neq g_2(x)$.

(Observe that when $x \in A$, then $g_1(x) = g_2(x) = f(x)$ since $g_1|_A = g_2|_A = f$.)

Since Y is Hausdorff and $g_1(x) \neq g_2(x)$, we have that there exist open nbhds U of $g_1(x)$ and V of $g_2(x)$ s.t. $U \cap V = \emptyset$.

Since g_1, g_2 are continuous and U, V are open in Y, we have that $g_1^{-1}(U)$ and $g_2^{-1}(V)$ are open in $\bar{A}$.

Observe that $x \in g_1^{-1}(U)$ and $x \in g_2^{-1}(V)$.

Let $W := g_1^{-1}(U) \cap g_2^{-1}(V)$. So $x \in W$, i.e., W is nonempty.

W is open since the finite intersection of open sets is open.

Observe that W is an open nbhd of x, and $x \in \bar{A}$.

Since $x \in \bar{A}$, we have that every open nbhd of x must intersect A.

Let $y \in g_1^{-1}(U) \cap g_2^{-1}(V) = W$.

$$\left.\begin{array}{l} \text{Since } y \in g_1^{-1}(U) \Rightarrow g_1(y) \in U \\ \text{Since } y \in g_2^{-1}(V) \Rightarrow g_2(y) \in V \end{array}\right\} \begin{array}{l} U \cap V = \emptyset \\ \text{So } g_1(y) \neq g_2(y) \ \forall y \in W. \end{array}$$

This means that $W \cap A = \emptyset$ since if $y \in A$, then $g_1(y) = g_2(y)$. ↯

This is a contradiction to x being a limit point.

Therefore, if there is a continuous extension of f to $\bar{A}$, then the extension is unique.

□

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③ Let $X = \{(x,y) \in \mathbb{R}^2 ; y = \pm 1\}$. Define M to be the quotient of X by the equivalence relation generated by $(x,1) \sim (x,-1)$ for all $x \neq 0$. Show that M is not Hausdorff.

Pf: Let $q: X \to X/\sim = M$ be the quotient map.

Consider $(0,1), (0,-1) \in M$.

We WTS that every open nbhd of $(0,1)$ intersects with every open nbhd of $(0,-1)$, i.e., M is not Hausdorff.

Let $U \subseteq M$ be an open nbhd of $(0,1)$.

Since q is continuous and U is open, we have that $q^{-1}(U)$ is open in X.

We know that $(0,1) \in q^{-1}(U)$, and since $q^{-1}(U)$ is open, we have

$(0,1) \in \{(x,1) : |x| < a\} \subseteq q^{-1}(U)$
and $\{(x,-1) : 0 < |x| < a\} \subseteq q^{-1}(U)$ } b/c $q^{-1}(U)$ is closed under the equivalence relation

Let $V \subseteq M$ be an open nbhd of $(0,-1)$.

Since q is continuous and V is open, we have that $q^{-1}(V)$ is open in X.

We know that $(0,-1) \in q^{-1}(V)$, and since $q^{-1}(V)$ is open, we have

$(0,-1) \in \{(x,-1) : |x| < b\} \subseteq q^{-1}(V)$
and $\{(x,1) : 0 < |x| < b\} \subseteq q^{-1}(V)$. } b/c $q^{-1}(V)$ is closed under the equivalence relation

Let $0 < c < \min\{a,b\}$.

Then $(c,1) \in q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V)$.

If $U \cap V = \emptyset$, then $q^{-1}(U \cap V) = q^{-1}(\emptyset) = \emptyset$.

Since $(c,1) \in q^{-1}(U \cap V)$, we have that $q^{-1}(U \cap V) \neq \emptyset$, so $U \cap V \neq \emptyset$.

Therefore, we have shown that every open nbhd of $(0,1)$ intersects with every open nbhd of $(0,-1)$.

Thus, M is not Hausdorff.

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④ Let $\mathbb{RP}^n$ be the real projective space, i.e., the quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ under the equivalence relation: $(x_0,\dots,x_n)\sim(y_0,\dots,y_n)$ if $(x_0,\dots,x_n)=\lambda(y_0,\dots,y_n)$ for some $\lambda\in\mathbb{R}$. Prove that the quotient map $q:\mathbb{R}^{n+1}\setminus\{0\}\to\mathbb{RP}^n$ is open.

Pf: We WTS that if $U\subseteq\mathbb{R}^{n+1}\setminus\{0\}$ is open, then $q(U)\subseteq\mathbb{RP}^n$ is open.

Since q is a quotient map, we know that $q(U)\subseteq\mathbb{RP}^n$ is open iff $q^{-1}(q(U))\subseteq\mathbb{R}^{n+1}\setminus\{0\}$ is open.

Therefore, to show that $q(U)$ is open in $\mathbb{RP}^n$, it suffices to show that $q^{-1}(q(U))$ is open in $\mathbb{R}^{n+1}\setminus\{0\}$.

- Let $U\subseteq\mathbb{R}^{n+1}\setminus\{0\}$ be open.

Let $x\in q^{-1}(q(U))$ (where x does not necessarily need to be in U).

We WTS that for $x\in q^{-1}(q(U))$, $\exists$ an open nbhd V of x s.t. $x\in V\subseteq q^{-1}(q(U))$

Since $x\in q^{-1}(q(U))$, we have that $q(x)\in q(U)$.

This means that there must be $y\in U$ s.t. $q(y)=q(x)$.

Then $x\sim y$, i.e., $y=\lambda x$ for some $\lambda\in\mathbb{R}$, $\lambda\neq 0$.

Since U is open and $y\in U$, we have $y\in B\subseteq U$, where B is an open ball around y (so $y=\lambda x\in B$).

We can rewrite $y=\lambda x$ as $x=\frac{1}{\lambda}y$.

So $x\in\frac{1}{\lambda}B=\lambda^{-1}B$, which is open because multiplying by a scalar is a homeomorphism, and homeo.'s send open sets to open sets.

Observe that $\{\frac{b}{\lambda}: b\in B\}\subseteq\frac{1}{\lambda}B \Rightarrow b\in\underbrace{q^{-1}(q(U))}_{\text{closed under equiv. relation.}}$

So $\frac{b}{\lambda}\in\underbrace{q^{-1}(q(U))}_{\text{contains everything related}\sim\text{to } b.}\ \forall b\in B\Rightarrow x\in\frac{1}{\lambda}B\subseteq q^{-1}(q(U))$ (where $\frac{1}{\lambda}B$ is open)

Therefore, $q^{-1}(q(U))$ is open in $\mathbb{R}^{n+1}\setminus\{0\}$.

Thus, $q(U)$ is open in $\mathbb{RP}^n$.

We conclude that q is open, i.e., if $U\subseteq\mathbb{R}^{n+1}\setminus\{0\}$ is open, then $q(U)\subseteq\mathbb{RP}^n$ is open.

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(5) Recall the diameter of a nonempty subset E of a metric space (M,d) is defined to be $diam(E) = \sup\{d(x,y); x,y \in E\}$. Show that the metric space (M,d) is complete if and only if the following property holds:
For any sequence $\{E_k\}_{k=1}^{\infty}$ of closed nonempty subsets of M satisfying $E_{k+1} \subset E_k$ for all $k \geq 1$, and $\lim_{k\to+\infty} diam(E_k) = 0$, the set $\bigcap_{k=1}^{\infty} E_k$ consists of precisely one point.

Pf: · Suppose that (M,d) is a complete metric space.
We WTS that $\bigcap_{k=1}^{\infty} E_k$ contains exactly one point. $(E = \bigcap_{k=1}^{\infty} E_k)$

· First we will show that $\bigcap_{k=1}^{\infty} E_k$ has at most one point (uniqueness).
Assume $\exists\, x,y \in E$ s.t. x,y are distinct. Then $d(x,y) > 0$.
We have that $diam(E_k) \geq d(x,y)$ $\forall i$ because $x,y \in E_k$ $\forall k \geq 1$.
So $0 = \lim_{k\to+\infty} diam(E_k) \geq d(x,y) > 0$. ↯ contradiction b/c $d(x,y) \neq 0$.
Therefore, $\bigcap_{k=1}^{\infty} E_k$ must contain at most one point.

· Now we will show that $\bigcap_{k=1}^{\infty} E_k$ has at least one point (existence).
Completeness implies that every Cauchy sequence converges.
We want to construct a Cauchy sequence of points.
Let $\{x_k\}_{k=1}^{\infty}$ be any sequence with $x_k \in E_k$.
Fix $\varepsilon > 0$. Let N be such that $diam(E_k) < \varepsilon$ $\forall k > N \in \mathbb{N}$.
Let $i,j \in \mathbb{N}$ s.t. $i > j$. Then the nested condition tells us that $E_i \subset E_j$, so $x_i, x_j \in E_j$.
So we have that $d(x_i, x_j) \leq diam(E_j) < \varepsilon$ as long as $j > N$.
For $i,j > N$, $\min(i,j) > N$ implies $d(x_i, x_j) < \varepsilon$.
Therefore, $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence.
Since $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, we have that $x_k \to x$ for some $x \in X$ because (M,d) is complete. We WTS that $x \in E$.
Observe that E_1: $\{x_k\}_{k=1}^{\infty} \subseteq E_1$, so closedness of E_1 tells us $x \in E_1$.
E_n: $\{x_k\}_{k=n}^{\infty} \subseteq E_n$, $\lim_{k\to+\infty} x_k = x$, so $x \in E_n$ by closedness.
Therefore, $x \in E_n$ $\forall n$ implies that $x \in E$.

Thus, since $\bigcap_{k=1}^{\infty} E_k$ contains at most and at least one point, it must consist of precisely one point. $\square$

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- Suppose the given property in the problem statement holds.

We WTS that (M,d) is complete.

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary Cauchy sequence in (M,d).

Define $E_K := \overline{\{x_n : n \geq K\}}$.

Observe that each E_K is closed, nonempty, and $E_{K+1} \subset E_K$.

To check the diameter condition, fix $\varepsilon > 0$.

Let N be such that $n, m \geq N$ implies $d(x_n, x_m) < \varepsilon$ (here we use the Cauchy property).

Then $\mathrm{diam}(E_N) = \mathrm{diam}(\{x_n : n \geq N\}) = \sup_{m,n > N} d(x_n, x_m) < \frac{\varepsilon}{2} < \varepsilon$, where we have used the fact that a set and its closure have the same diameter.

$(\mathrm{diam}(E_K) \to 0)$

Since we have checked all the conditions, the given property tells us that $\bigcap_{K=1}^{\infty} E_K$ is a one-point set.

Let $\bigcap_{K=1}^{\infty} E_K = \{x\}$. We want to check that $x_K \to x$.

Fix $\varepsilon > 0$. The ball $B(x, \varepsilon/2)$ is a nbhd of x; since $x \in E_K$, $B(x, \varepsilon/2)$ intersects $\{x_n : n \geq K\}$ for every K.

Choose N large enough that $d(x_n, x_m) < \varepsilon/2$ whenever $n, m > N$.

Let $k > N$ be any index such that $x_k \in B(x, \varepsilon/2)$.

Now if $j > N$ is arbitrary, we see

$$d(x, x_j) \leq d(x, x_k) + d(x_k, x_j) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $B(x, \frac{\varepsilon}{2})$ contains a tail end of the sequence. Hence $x_n \to x$.

Therefore, (M,d) is complete.

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⑥ Let X be a topological space.

(i) Show that if X is compact Hausdorff, then X is normal.

Pf: Let $E_1, E_2 \subseteq X$ be disjoint closed subsets of X.

We WTS that $\exists$ open nbhds U of E_1 and V of E_2 s.t. $U \cap V = \emptyset$.

- Observe that for every $x \in E_1$ and $y \in E_2$, $\exists$ open nbhds U of x and V of y s.t. $U \cap V = \emptyset$ because X is Hausdorff.
- Now fix $x \in X$ and $E_1 \subseteq X$ closed s.t. $x \notin E_1$.

For each $y \in E_1$, we have open nbhds U_y of x and V_y of y s.t. $U_y \cap V_y = \emptyset$.

Observe that E_1 is compact since it is a closed subset of a compact space.

Since E_1 is compact, we have that for a given open cover $\{V_y : y \in E_1\}$, there is a finite subcover $\{V_{y_i} : y_i \in E_1, 1 \le i \le n\}$.

Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$.

Observe that $V \neq \emptyset$ because $E_1 \subseteq V$, and $U \neq \emptyset$ b/c $x \in U$.

We also have that V is open b/c the union of arbitrarily many open sets is open, and U is open b/c the finite intersection of open sets is open.

So we have that U is an open nbhd of x and V is an open nbhd of E_1 s.t. $U \cap V = \emptyset$ b/c $U_y \cap V_y = \emptyset$ $\forall y \in E_1$.

Therefore, we have shown that X is regular.

- For each $x \in E_1$, we have open nbhds U_x of x and V_x of E_2 s.t. $U_x \cap V_x = \emptyset$ because X is regular.

Observe that E_1 is compact since it is a closed subset of a compact space.

Since E_1 is compact, we have that for a given open cover $\{U_x : x \in E_1\}$, there is a finite subcover $\{U_{x_i} : x_i \in E_1, 1 \le i \le n\}$.

Let $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcap_{i=1}^{n} V_{x_i}$.

Observe that $U \neq \emptyset$ b/c $E_1 \subseteq U$ and $V \neq \emptyset$ b/c $y \in V$ where $y \in E_2$ ($E_2 \subseteq V$).

We also have that U is open b/c the union of arbitrarily many open sets is open, and V is open b/c the finite intersection of open sets is open.

So U is an open nbhd of E_1 and V is an open nbhd of E_2 s.t. $U \cap V = \emptyset$ b/c $U_x \cap V_x = \emptyset$ $\forall x \in E_1$.

Therefore, if X is compact Hausdorff, then X is normal.

$\square$

continued...

(ii) In class we proved that if X is second countable and regular, then it is normal. Assume now X is secound countable and Hausdorff. Is it true that X is normal? Prove this statement if true. otherwise, provide a counterexample.

Pf: Consider $\mathbb{R}_K$.

- Let $\mathcal{B} = \{(a,b), (a,b) \setminus K\}$ be a basis for $\mathbb{R}_K$, where $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$.
  Observe that $\mathcal{B}$ with $a, b \in \mathbb{Q}$ is a countable basis (similar to the argument that $\{(a,b) : a, b \in \mathbb{Q}\}$ forms a basis of $\mathbb{R}$). So $\mathbb{R}_K$ is second countable.
- Let $x, y \in \mathbb{R}_K$ s.t. $x \neq y$. Then there exist open nbhds U of x and V of y s.t. $U \cap V = \emptyset$.
  Let $x, y \in \mathbb{R}$ s.t. $x \neq y$. We have that $\mathbb{R}_S$ is Hausdorff, so $\exists$ open nbhds U of x and V of y s.t. $U \cap V = \emptyset$.
  Since $\mathbb{R}_K$ contains more open sets than $\mathbb{R}_S$, we have that $\mathbb{R}_K$ is Hausdorff since $\mathbb{R}_S$ is.

Therefore, $\mathbb{R}_K$ is second countable and Hausdorff.

- It suffices to show that $\mathbb{R}_K$ is not regular.
  We WTS that $\exists\, x \in \mathbb{R}_K$ and a closed set $E \subseteq \mathbb{R}_K$ s.t. every open nbhd U of x and V containing E have $U \cap V \neq \emptyset$.

  Let $x = 0 \in \mathbb{R}_K$ and let $E = K$.

  To show that $E = K$ is closed in $\mathbb{R}_K$, we will show that $\mathbb{R} \setminus K$ is open.

  Observe that $\mathbb{R} \setminus K = \bigcup_{n=1}^{\infty} (-n, n) \setminus K$, so $E = K$ is closed.

  Let U be an open nbhd of 0 and V an open nbhd of E.

  Then $B_r(0) \setminus K \subseteq U$.

  U contains some arbitrarily small irrational.

  E contains $\frac{1}{n}$ for some $n \in \mathbb{Z}^+$ s.t. $\frac{1}{n} < \frac{r}{2}$.

  V contains an irrational number $\varepsilon < \frac{1}{n} < \frac{r}{2}$.

  So $\varepsilon \in B_r(0) \setminus K \Rightarrow \varepsilon \in U \cap V$.

  Therefore, $U \cap V \neq \emptyset \Rightarrow X$ is not regular.

  Thus, X is not normal.

$\square$