~~Topotogy~~ HW3

① Let $A = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y \in \mathbb{Q}\}$, $B = \{(x,y) \in \mathbb{R}^2 : \text{at least one of } x, y \text{ is rational}\}$.
Is the space $\mathbb{R}^2 \setminus A$ connected? Is the space $\mathbb{R}^2 \setminus B$ connected? Prove your assertions.

Pf: • First we will show that $\mathbb{R}^2 \setminus A$ is connected.
Observe that $\mathbb{R}^2 \setminus A$ consists of coordinates $(x,y) \in \mathbb{R}^2$ where either both $x, y$ are irrational or exactly one of $x, y$ is rational.
Let $(x,y), (w,z) \in \mathbb{R}^2 \setminus A$.
Then either $x$ or $y$ is irrational and either $w$ or $z$ is irrational.
WLOG, suppose $x$ is irrational.
We WTS there exists a path from $(x,y)$ to $(w,z)$ that avoids $A$.
Since $x$ is irrational, there exists a straight line path from $(x,y)$ to $(x,u)$ where $u$ is irrational.
If $w$ is irrational, then there exists a straight line path from $(x,u)$ to $(w,u)$, and then a straight line path from $(w,u)$ to $(w,z)$.
If $z$ is irrational, then there exists a straight line path from $(x,u)$ to $(x,z)$, and then a straight line path from $(x,z)$ to $(w,z)$.
Therefore, we can construct a path between any two points in $\mathbb{R}^2 \setminus A$.
Thus, $\mathbb{R}^2 \setminus A$ is path-connected $\Rightarrow \mathbb{R}^2 \setminus A$ is connected.

• Now we will show that $\mathbb{R}^2 \setminus B$ is not connected.
Observe that $\mathbb{R}^2 \setminus B$ consists of coordinates $(x,y) \in \mathbb{R}^2$ where both $x, y$ are irrational.
Notice that $\mathbb{R}^2 \setminus B = \{(x,y) \in \mathbb{R}^2 \setminus B : x < 0\} \cup \{(x,y) \in \mathbb{R}^2 \setminus B : x > 0\}$ since there is no point of the form $(0,b)$, $b \in \mathbb{R}$, because $0$ is rational.
We have that $\{(x,y) \in \mathbb{R}^2 \setminus B : x < 0\} =: U$ and $\{(x,y) \in \mathbb{R}^2 \setminus B : x > 0\} =: V$ are nonempty, open, and disjoint (i.e. $U \cap V = \emptyset$).
Therefore, $\mathbb{R}^2 \setminus B = U \cup V$ is a separation of $\mathbb{R}^2 \setminus B$.
Thus, $\mathbb{R}^2 \setminus B$ is not connected.

$\square$

continued...

② Let $\mathbb{RP}^2$ be the real projective plane, that is, the topological space of lines in $\mathbb{R}^3$ passing through the origin. One construction of $\mathbb{RP}^2$ is as a quotient space of the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ with the subspace topology, obtained by identifying antipodal points. Prove that $\mathbb{RP}^2$ is compact Hausdorff, and that every point in $\mathbb{RP}^2$ has a nbhd homeomorphic to the open unit ball in $\mathbb{R}^2$.

Pf: Let $q: S^2 \to \mathbb{RP}^2$ be the quotient map.

- First we will show that $\mathbb{RP}^2$ is compact.
  We know that $S^2$ is closed and bounded as a subset of $\mathbb{R}^n$, therefore, $S^2$ is compact. Since the continuous image of a compact space is compact, we have that $q(S^2)$ is compact.
  Since $q$ is a quotient map, it is surjective, so we also know that $q(S^2) = \mathbb{RP}^2$. Therefore, $\mathbb{RP}^2$ is compact.
- Notice that $q$ is an open map.
  To see this, notice that if $U$ is open, then $q^{-1}(q(U)) = U \cup (-U)$, and the antipode map is a homeomorphism of $S^2$, so $-U$ is open. $\Rightarrow U \cup (-U)$ is open.
  Hence, $q^{-1}(q(U))$ is open, so by defn. of a quotient map, $q(U)$ is open.
- To show that $\mathbb{RP}^2$ is Hausdorff, let $[x],[y]$ be distinct points of $\mathbb{RP}^2$.
  Let $r = \min\{d(x,y)/4,\ d(x,-y)/4,\ 1/4\}$.
  Then the four sets $B(x,r), B(y,r), B(-x,r), B(-y,r)$ are disjoint.
  If two of the balls intersect at a point $z$, this immediately implies the distance between their centers is at most $2r$, but $2r$ is less than the distance between any of their centers by our choice of $r$.
  This means that no point of $B(x,r)$ is even related to a point of $B(y,r)$.
  In other words, if $z \in B(x,r)$, then $q(z) \notin q(B(y,r))$.
  This guarantees that $q(B(x,r))$ is disjoint from $q(B(y,r))$.
  Since $q$ is an open map, we have that $q(B(x,r))$ and $q(B(y,r))$ are open.
  Therefore, $\mathbb{RP}^2$ is Hausdorff.

$\square$

...ed...

Let $f: X \to Y$ be a continuous and injective map between topological spaces $X$ and $Y$. Prove that if $X$ is compact and $Y$ is Hausdorff, then $f$ is an embedding.

Pf: Since we are given that $f$ is continuous and injective, it suffices to show that $f$ is a closed map.

Let $K \subseteq X$ be closed.

A closed subset of a compact space is compact.

Since $K$ is closed, $X$ is compact, and $K \subseteq X$, we have that $K$ is compact.

The continuous image of a compact space is compact.

Since $f$ is continuous and $K$ is compact, we have that $f(K) \subseteq Y$ is compact.

A compact subset of a Hausdorff space is closed.

Since $f(K)$ is compact, $Y$ Hausdorff, and $f(K) \subseteq Y$, we have that $f(K)$ is closed in $Y$.

Therefore, if $K$ is closed in $X$, then $f(K)$ is closed in $Y$.

So we have that $f$ is a closed map.

Thus, $f$ is an embedding. $\square$

...ued...

Let $B = \{x \in \mathbb{R}^2;\ |x| < 1\}$. Let $\{f_k\}$ be a sequence of continuous functions on $B$ satisfying $\sup_{x\in B} |f_k(x)| + \sup_{\substack{x \neq y \\ x,y \in B}} \dfrac{|f_k(x) - f_k(y)|}{|x-y|^{1/2}} \leq C$ for some positive constant $C$.

Show that there is a subsequence of $\{f_k\}$ convergent uniformly on $B$ to a function $g$, and $g$ is continuous on $B$.

Pf: Recall the following version of Arzela-Ascoli:

Theorem: Let $X$ be a compact metric space and $\{g_k\}$ a sequence that is uniformly bounded and uniformly equicontinuous. Then there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k}$ converges uniformly.

- Notice that the assumptions that we've been given in this problem imply that
  - (1) The family $\{f_k : k \in \mathbb{N}\}$ is uniformly bounded.
  - (2) The family $\{f_k : k \in \mathbb{N}\}$ is uniformly equicontinuous.
  - (3) Each $f_k$ is uniformly continuous on $B$.

We can almost apply Arzela-Ascoli, except that $B$ is not a compact metric space. However, if we let $D$ denote the closed unit ball in $\mathbb{R}^2$, condition (3) says that each $f_k$ is uniformly continuous on $B$, a dense subset of $D$.

- Recall the following extension theorem:

Theorem: Let $X, Y$ be metric spaces with $Y$ complete. If $D$ is a dense subset of $X$ and $f: D \to Y$ is uniformly continuous, then $f$ has a continuous extension to $X$.

- This theorem implies that each $f_k$ extends to a continuous function $\tilde{f}_k: D \to \mathbb{R}$. Continuity of each $\tilde{f}$ implies that the estimate

$$\sup_{x\in D} |\tilde{f}_k(x)| + \sup_{\substack{x \neq y \\ x,y \in D}} \frac{|\tilde{f}_k(x) - \tilde{f}_k(y)|}{|x-y|^{1/2}} \leq C \text{ holds.}$$

Hence conditions (1) and (2) still hold with $\tilde{f}_k$ in place of $f_k$, and $D$ actually is a compact metric space, so Arzela-Ascoli gives us a subseq. $\tilde{f}_{n_k}$ that converges uniformly to a limit function $F: D \to \mathbb{R}$. Notice that

$$\sup_{x\in B} |f_{n_k}(x) - F(x)| = \sup_{x \in B} |\tilde{f}_{n_k}(x) - F(x)| \leq \sup_{x\in D} |\tilde{f}_{n_k}(x) - F(x)| \to 0.$$

So $f_{n_k}$ converges uniformly to $g := F|_B$. The uniform limit of continuous functions is continuous, so $g$ is continuous. $\square$

continued...

⑥ Let $Mat_3(\mathbb{R})$ be the set of $3\times 3$ matrices with the topology obtained by regarding $Mat_3(\mathbb{R})$ as $\mathbb{R}^9$. Let $SO(3)=\{A\in Mat_3(\mathbb{R}); A^TA=I_3, \det(A)=1\}$, where $A^T$ denotes the transpose of A, and $I_3$ is the $3\times 3$ identity matrix.

(i) Show that $SO(3)$ is compact.

Pf: By Heine-Borel it suffices to show $SO(3)$ is closed and bounded.

To show closedness, notice that the map $g: Mat_3(\mathbb{R})\to Mat_3(\mathbb{R})$ defined by $g(A)=A^TA$ is continuous, so $g^{-1}(\{I_3\})$ is closed (because one-point sets are closed in $\mathbb{R}^9$).

Similarly, the determinant map is continuous, so $\{A\in Mat_3(\mathbb{R}): \det(A)=1\}$ is closed. Hence, their intersection, $SO(3)$, is a closed set.

($SO(3)=g^{-1}(\{I_3\})\cap \det^{-1}(\{1\})$ → closed as the intersection of closed sets)

To see boundedness, notice that

$$\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix} a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Performing the multiplication on the right and equating diagonal entires gives us

$$a^2+b^2+c^2=1$$
$$d^2+e^2+f^2=1$$
$$g^2+h^2+i^2=1$$

, so that if $A\in SO(3)$, we have $\|A\|^2=a^2+b^2+\dots+i^2=3$, so that $A\in B(0,\sqrt{3})$.

So if $A^TA=I$, then $A\in B(0,\sqrt{3})$.

By Heine-Borel, $SO(3)$ is compact. □

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Let $S = \{B \in SO(3);\ B^T = B\}$. Define $\varphi : \mathbb{RP}^2 \to SO(3)$ by $\varphi(\ell)$ = the rotation by $\pi$ about the line $\ell \subset \mathbb{R}^3$.
Show that $\varphi$ maps $\mathbb{RP}^2$ homeomorphically onto $S \setminus \{I_3\}$.

Pf.
- The map $\varphi$ is continuous, because the coordinates of the rotation matrix about the line $\ell$ depend continuously on the coordinates of the points $\ell \cap S^2$ (i.e., if the equation of $\ell$ is $t\langle a,b,c\rangle$, then the rotation about $\ell$ depends on $\langle a,b,c\rangle$.)
- The map $\varphi$ is injective, because the rotation by $\pi$ about $\ell_1$ is distinct from rotation by $\pi$ about $\ell_2$ whenever $\ell_1$ and $\ell_2$ are distinct lines.
- The map $\varphi$ is surjective. Recall that every rotation matrix satisfies $A^TA = I$, so in particular $A^T = A^{-1}$. The only rotations that are self-inverse are rotations by $0$ and rotations by $\pi$. $I_3$ is not in the codomain of $\varphi$, so every element of the codomain is a rotation by $\pi$.
- Finally $\varphi$ is closed, because it is a continuous map with a compact domain ($\mathbb{RP}^2$) and Hausdorff codomain ($SO(3)$).
  $\hookrightarrow S \setminus \{I_3\} \subseteq \mathbb{R}^9 \to$ Hausdorff.

Therefore, we conclude that $\varphi : \mathbb{RP}^2 \to S \setminus \{I_3\}$ is a homeomorphism. $\square$