Topology HW 4

① Let X be a topological space and $f, g: X \to S^2$ two continuous maps. Show that if for every $x \in X$ the points $f(x)$ and $g(x)$ on S^2 are not antipodal to each other, then f and g are homotopic.

Pf: Let $H: [0,1] \times X \to S^2$ be given by $H(t,x) = \dfrac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$.

We WTS that H is a homotopy.

Observe that H is continuous because it is the product and sum of continuous functions.

First we will check that H is well-defined (i.e., $(1-t)f(x) + tg(x) \neq 0$):

$$(1-t)f(x) + tg(x) = 0 \Rightarrow (1-t)f(x) = -tg(x)$$

$$\|(1-t)f(x)\| = \|-tg(x)\| \quad \text{because } \|f(x)\| = 1,\ \|g(x)\| = 1$$

$$\|(1-t)\| = \|-t\|$$

$$1-t = t$$

$$1 = 2t$$

$$\Rightarrow t = \tfrac{1}{2}$$

At $t = \frac{1}{2}$, we have: $\left(1 - \frac{1}{2}\right) f(x) = -\frac{1}{2} g(x)$

$$\tfrac{1}{2} f(x) = -\tfrac{1}{2} g(x)$$

$$f(x) = -g(x),$$

which cannot happen because $f(x)$ and $g(x)$ on S^2 are not antipodal to each other. So we have that $(1-t)f(x) + tg(x) \neq 0$.

Therefore, H is well-defined.

Now we will check that H is a free homotopy:

$$H(0,x) = \frac{(1-0)f(x) + 0 \cdot g(x)}{\|(1-0)f(x) + 0 \cdot g(x)\|} = \frac{f(x)}{\|f(x)\|} = f(x),$$

$$H(1,x) = \frac{(1-1)f(x) + 1 \cdot g(x)}{\|(1-1)f(x) + 1 \cdot g(x)\|} = \frac{g(x)}{\|g(x)\|} = g(x).$$

Therefore, H is a homotopy.

Thus, we conclude that f and g are homotopic. $\square$

continued...

(2) Is the following statement true? A space X is contractible if and only if every map $f: X \to Y$ continuous, for an arbitrary Y, is null-homotopic. Prove your assertion.

Pf: Recall that a space X is contractible if Id_X is freely homotopic to a constant map, i.e., null-homotopic.

- Suppose every map $f: X \to Y$ continuous, for an arbitrary Y, is null-homotopic.
  Let $Y = X$, so $f: X \to X$ s.t. $f(x) = x$, i.e., f is the identity map, Id_X.
  Then by our supposition, we have that $f = Id_X$ is null-homotopic.
  Therefore, X is contractible (by definition).

- Suppose X is contractible.
  Since X is contractible, we have that Id_X is homotopic to a constant map c_{x_0}. Let $H: [0,1] \times X \to X$ be the homotopy between Id_X and c_{x_0}, so we have $H(0,x) = Id_X(x) = x$ and
  $H(1,x) = c_{x_0}(x) = x_0$ constant.

  Let $\tilde{H}: [0,1] \times X \to Y$ be defined by $\tilde{H}(t,x) = (f \circ H)(t,x)$.
  We WTS that $\tilde{H}$ is the homotopy of f to a constant map.
  Observe that $\tilde{H}$ is continuous because it is the composition of continuous functions.
  Observe that $\tilde{H}(0,x) = f(H(0,x)) = f(x)$ and
  $\tilde{H}(1,x) = f(H(1,x)) = f(x_0)$ constant.
  Therefore, we conclude that $\tilde{H}$ is a homotopy, i.e., f is freely homotopic to a constant map at $f(x_0)$.
  Thus, every map $f: X \to Y$ continuous, for an arbitrary Y, is null-homotopic.

$\square$

...ued...

A space X is called a strong deformation retraction to a point $x_0 \in X$ if there is a continuous map $F: [0,1] \times X \to X$ such that $F(0,x) = x$, $F(1,x) = x_0$ for all $x \in X$ and $F(t, x_0) = x_0$ for all $0 \le t \le 1$.

Show that X is simply-connected if X is a strong deformation retraction to a point.

Pf: Suppose that X is a strong deformation retraction to a point.
We WTS that X is simply-connected, i.e., X is path-connected and that $\pi_1(X, x_0)$ is trivial for some $x_0 \in X$.

- First we will show that X is path-connected.
  Let $\alpha: [0,1] \to X$ by $\alpha(t) = F(t,x)$.
  Observe that α is continuous since F is continuous.
  Observe that $\alpha(0) = F(0,x) = x$ and $\alpha(1) = F(1,x) = x_0$.
  Therefore, we have shown that α is a path.
  We can defined such a path $\forall x \in X$.
  Thus, X is path-connected.
- Now we will show that $\pi_1(X, x_0)$ is trivial.
  We will do this by taking an arbitrary loop based at x_0 and showing that it is path-homotopic to a constant map.
  Let $\beta: [0,1] \to X$ such that $\beta(0) = \beta(1) = x_0$ be an arbitrary loop based at x_0.
  Define $H: [0,1] \times [0,1] \to X$ by $H(t,s) = F(t, \beta(s))$.
  It is clear that H is continuous.
  Observe that
  $$\begin{aligned} H(0,s) &= F(0, \beta(s)) = \beta(s) \\ H(1,s) &= F(1, \beta(s)) = x_0 \\ H(t,0) &= F(t, \beta(0)) = F(t, x_0) = x_0 \\ H(t,1) &= F(t, \beta(1)) = F(t, x_0) = x_0. \end{aligned}$$

  Therefore, H is a path-homotopy.
  Thus, we have shown that any arbitrary loop based at x_0 is path-homotopic to the constant map at x_0, so there is only one element in the fundamental group. We conclude that $\pi_1(X, x_0)$ is trivial.

Since X is path-connected and $\pi_1(X, x_0)$ is trivial, it follows that X is simply-connected.

$\square$

continued...

(4) Let A be a subset of a topological space X. Suppose that $r: X \to A$ is a retraction of X onto A, i.e., r is a continuous map such that the restriction of r to A is the identity map of A.

(1) Show that if X is Hausdorff, then A is a closed subset of X.

Pf: Let $A = \{x \in X : r(x) = x\}$. We WTS that A is closed.

We will do this by showing that $X \setminus A = \{x \in X : r(x) \neq x\}$ is open.

Let $x \in X \setminus A$. Then we have $r(x), x \in X$ s.t. $r(x) \neq x$.

Since X is Hausdorff and $r(x) \neq x$, $\exists$ open nbhds U of $r(x)$ and V of x s.t. $U \cap V = \emptyset$.

Since $r(x) \in U$, we have that $x \in r^{-1}(U)$, which is open since r is cts.

Let $W := r^{-1}(U) \cap V$.

We have that W is open since the finite intersection of open sets is open.

We also have that W is nonempty since $x \in r^{-1}(U)$ and $x \in V$, so $x \in W$.

So W is an open nbhd of x.

We WTS that $W \cap A = \emptyset$:

Let $y \in r^{-1}(U) \cap V = W$.

$\left.\begin{array}{l}\text{Since } y \in r^{-1}(U), \text{ we have that } r(y) \in U \\ \text{Since } y \in W,\ y \in V.\end{array}\right\}$ $U \cap V = \emptyset$, so $r(y) \neq y$ $\forall y \in W$.

Therefore, $W \cap A = \emptyset$.

Thus, we have $x \in \underset{\text{open}}{W} \subseteq X \setminus A \Rightarrow X \setminus A$ is open $\Rightarrow A$ is closed.

We conclude that A is a closed subset of X. $\square$

(2) Let $a \in A$. Show that $r_*: \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Pf: Let $a \in A$ and $\alpha \in \pi_1(A, a)$.

Recall that by definition, $r_*([\alpha]) = [r \circ \alpha]$.

Since α is a loop in A based at a and we know that the restriction of r to A is the identity map of A, we have that $r \circ \alpha = \alpha$.

Therefore, $r_*([\alpha]) = [r \circ \alpha] = [\alpha]$, so α is a loop in $\pi_1(X, a)$ such that $r_*([\alpha]) = [\alpha]$.

Thus, r_* is surjective. $\square$

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) Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map from X to the torus $\mathbb{T}^2$ is null-homotopic.

Pf: Let $f: X \to \mathbb{T}^2$ be a continuous map.

We want to use the general lifting lemma to show that there exists a lift $\tilde{f}: X \to \mathbb{R}^2$.

Observe that X is path-connected and locally path-connected.

Observe that $\mathbb{R}$ is the covering space of S^1 and the product of covering maps is a covering map, so $p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$.

Therefore, $p: \mathbb{R}^2 \to \mathbb{T}^2$ is a covering map.

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{T}^2 \end{tikzcd}$$

To use the general lifting lemma, it remains to show that $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}^2))$.

Observe that $\pi_1(\mathbb{R}^2) = \pi_1(\mathbb{R}) \times \pi_1(\mathbb{R}) = 0 \times 0 = 0$.

So we WTS that $f_*(\pi_1(X)) = 0$.

Observe that $\pi_1(X)$ is finite, so $f_*(\pi_1(X))$ is finite.

We have that $f_*(\pi_1(X)) \subseteq \pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

The only finite subgroup of $\mathbb{Z}$ is 0, so $f_*(\pi_1(X)) = 0 \subseteq p_*(\pi_1(\mathbb{R}^2))$ ✓

Therefore, by the general lifting lemma, we have that $\tilde{f}: X \to \mathbb{R}^2$ is a lift.

A continuous map into a contractible space is null-homotopic.

Since $\tilde{f}$ is continuous and $\mathbb{R}^2$ is contractible, we have that $\tilde{f}$ is null-homotopic.

Since $\tilde{f}$ is null-homotopic, we have that f is null-homotopic (if H is the homotopy between $\tilde{f}$ and a constant map, then $p \circ H$ is the homotopy between f and a constant map.).

Therefore, every map $f: X \to \mathbb{T}^2$ is null-homotopic. □

continued...

⑥ Prove that any continuous map from $\mathbb{RP}^2$ to S^1 is homotopic to a constant map. For problem 6, you can use the fact that the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$.

Pf: Let $f: \mathbb{RP}^2 \to S^1$ is a continuous map.
We want to use the general lifting lemma to show that there exists a lift $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}$.
We know that $\mathbb{RP}^2$ is path-connected because it is the continuous image of S^2, which is path-conn., and the cts image of a path-conn. space is path-conn.
We know that $\mathbb{RP}^2$ is locally path-connected because $q: S^2 \to \mathbb{RP}^2$ is a local homeomorphism.
We have that $p: \mathbb{R} \to S^1$ (the exp. map) is a covering map.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & S^1 \end{tikzcd}$$

In order to use the general lifting lemma, it remains to show that $f_*(\pi_1(\mathbb{RP}^2)) \subseteq p_*(\pi_1(\mathbb{R}))$.
Observe that $\pi_1(\mathbb{R}) = 0$ because $\mathbb{R}$ is contractible.
So we WTS that $f_*(\pi_1(\mathbb{RP}^2)) = 0$.
We have that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, so it is finite, and so $f_*(\pi_1(\mathbb{RP}^2))$ must be finite.
We have that $f_*(\pi_1(\mathbb{RP}^2)) \subseteq \pi_1(S^1) = \mathbb{Z}$.
The only finite subgroup of $\mathbb{Z}$ is 0, so $f_*(\pi_1(\mathbb{RP}^2)) = 0 \subseteq p_*(\pi_1(\mathbb{R}))$. ✓
Therefore, by the general lifting lemma, we have that $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}$ is a lift.
A continuous map into a contractible space is null-homotopic.
Since $\tilde{f}$ is continuous and $\mathbb{R}$ is contractible, we have that $\tilde{f}$ is null-homotopic.
Since $\tilde{f}$ is null-homotopic, we have that f is null-homotopic because if H is a homotopy between $\tilde{f}$ and a constant map, then $p \circ H$ is a homotopy between f and a constant map.
Therefore, $f: \mathbb{RP}^2 \to S^1$ is null-homotopic, i.e., homotopic to a constant map. □