Topology HW 5

① Compute the fundamental groups of the following spaces:

(1) The sphere with k distinct points removed, $k \geq 1$.

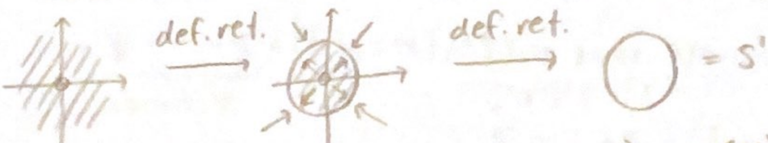
Pf: Via stereographic projection, we have that $S^2 \setminus \{k \text{ points}\}$ is homeo. to $\mathbb{R}^2 \setminus \{k-1 \text{ points}\}$.

We will induct on k.

Base case: $k=1$: Then $S^2 \setminus \{1 \text{ point}\} \xrightarrow{\text{stereo. proj}} \mathbb{R}^2 \setminus \{0 \text{ points}\} = \mathbb{R}^2$

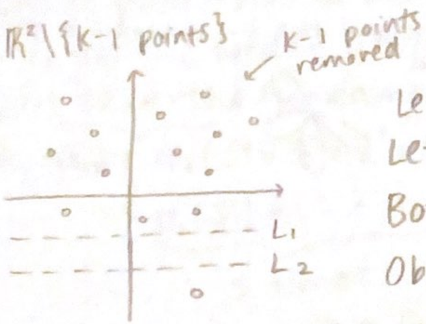
$\pi_1(\mathbb{R}^2) = 0 \Rightarrow \pi_1(S^2 \setminus \{1 \text{ point}\}) = 0.$

$k=2$: Then $S^2 \setminus \{2 \text{ points}\} \xrightarrow{\text{stereo. proj}} \mathbb{R}^2 \setminus \{1 \text{ point}\}$



$\pi_1(S^2 \setminus \{2 \text{ points}\}) = \pi_1(\mathbb{R}^2 \setminus \{1 \text{ point}\}) = \pi_1(S^1) = \mathbb{Z}.$

Induction hypothesis: Assume that $\pi_1(S^2 \setminus \{k-1 \text{ points}\}) = \pi_1(\mathbb{R}^2 \setminus \{k-2 \text{ points}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-2 \text{ copies}}$



Let U = everything strictly above L_2.

Let V = everything strictly below L_1.

Both U, V are path-connected and open.

Observe that $\mathbb{R}^2 \setminus \{k-2 \text{ points}\} = U \cup V$.

By the induction hypothesis, we have that $\pi_1(U) = \pi_1(\mathbb{R}^2 \setminus \{k-2 \text{ points}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-2 \text{ copies}}$

By the base case, we have that $\pi_1(V) = \pi_1(\mathbb{R}^2 \setminus \{1 \text{ point}\}) = \mathbb{Z}$.

Observe that $U \cap V$ is the open strip between L_1 and L_2. We have that $U \cap V$ is nonempty and path-connected. Since $U \cap V$ is contractible, we have that $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply-connected, we can use the following version of Van-Kampen: $\pi_1(\mathbb{R}^2 \setminus \{k-1 \text{ points}\}) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V)$

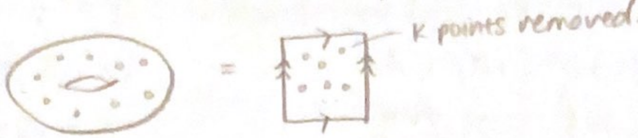
$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-2 \text{ times}} * \mathbb{Z} = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-1 \text{ times}}.$

Therefore, the fundamental group of S^2 with k distinct points removed ($k \geq 1$) is $\pi_1(S^2 \setminus \{k \text{ pts}\}) = \pi_1(\mathbb{R}^2 \setminus \{k-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k-1 \text{ times}}$. $\square$

continued...

(2) The torus $\mathbb{T}^2$ with k distinct points removed, $k \geq 1$.

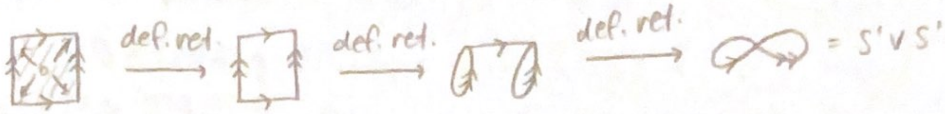
Pf: X



Let $X = \mathbb{T}^2 \setminus \{k \text{ points}\}$.

We will use induction to show that $\pi_1(\mathbb{T}^2 \setminus \{k \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k+1 \text{ times}}$

Base case: $k=1$:

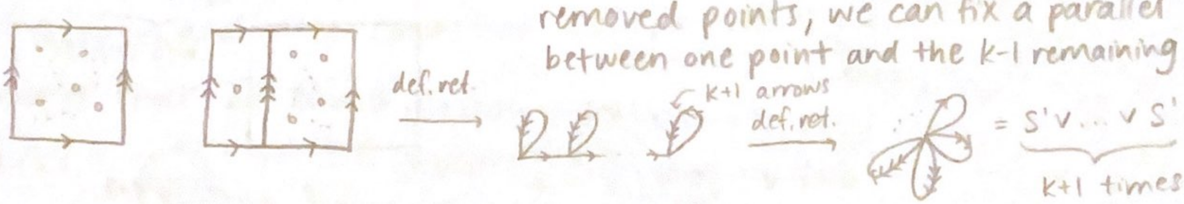


$\pi_1(\mathbb{T}^2 \setminus \{1 \text{ point}\}) = \pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$.

Induction hypothesis: Assume that $\pi_1(\mathbb{T}^2 \setminus \{k-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k \text{ times}}$

We WTS $\pi_1(\mathbb{T}^2 \setminus \{k \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k+1 \text{ times}}$

Let $\mathbb{T}^2 \setminus \{k \text{ pts}\}$ be drawn below. Since there is space between the removed points, we can fix a parallel line between one point and the $k-1$ remaining points.

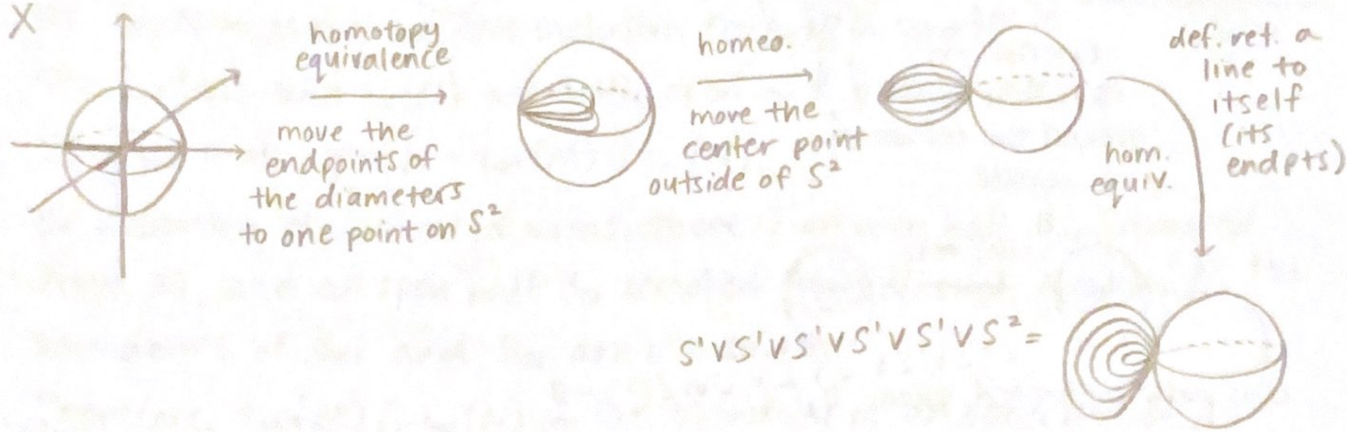


$$\pi_1(\mathbb{T}^2 \setminus \{k \text{ points}\}) = \pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{k+1 \text{ times}})$$
$$= \underbrace{\pi_1(S^1) * \dots * \pi_1(S^1)}_{k+1 \text{ times}}$$
$$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k+1 \text{ times}}$$

Therefore, $\pi_1(\mathbb{T}^2 \setminus \{k \text{ points}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k+1 \text{ times}}$. $\square$

Let X be the subspace of $\mathbb{R}^3$ equal to the union of the unit sphere with the three line segments $\{(x,0,0); |x| \le 1\} \cup \{(0,y,0); |y| \le 1\} \cup \{(0,0,z); |z| \le 1\}$. Compute the fundamental group of X based at $(1,0,0)$.

Pf: Since X is path-connected, the fundamental group is independent of the base point (up to isomorphism).



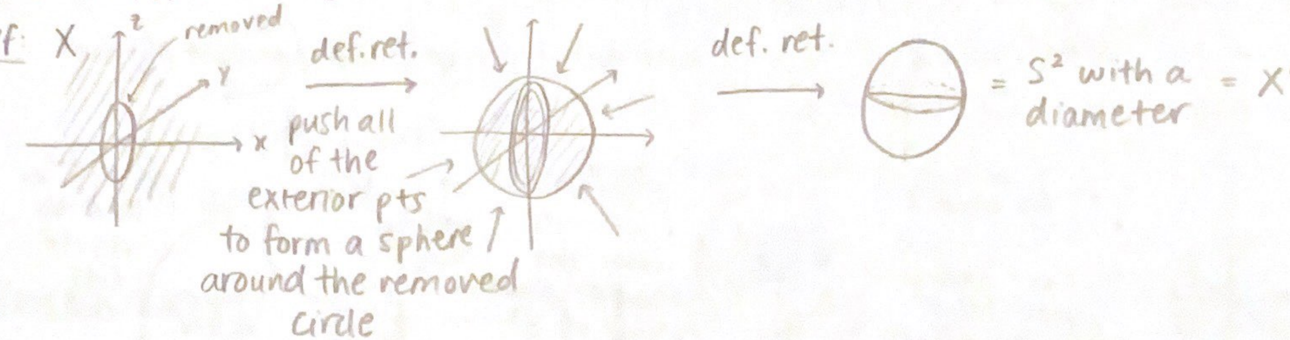
Each space, S^1 and S^2, is locally Euclidean, so the wedge point has a nbhd in each space that def. ret. to the wedge point, so using Van-Kampen, we can take the fundamental group as follows:

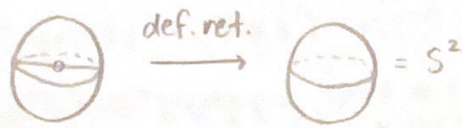
$$\pi_1(X) = \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^2)$$
$$= \underbrace{\pi_1(S^1) * \dots * \pi_1(S^1)}_{5 \text{ times}} * \pi_1(S^2)$$
$$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{5 \text{ times}} * 0$$

Therefore, $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. $\square$

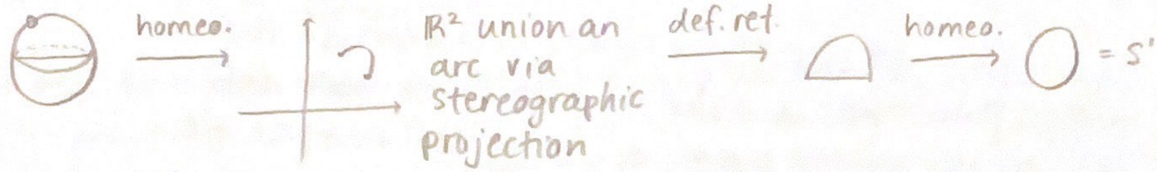
continued...

③ Let X be the space obtained from $\mathbb{R}^3$ by removing the circle $C = \{(0,y,z) \in \mathbb{R}^3 : y^2 + z^2 = 9\}$. Compute $\pi_1(X)$.

Pf: 

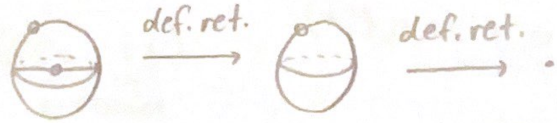
Let U = 

U is path-connected, open. $\pi_1(U) = \pi_1(S^2) = 0$.

Let V = 

V is path-connected, open. $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$.

Note that $X' = U \cup V$.

We have $U \cap V$ = 

$U \cap V$ is path-connected and nonempty. $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen:

$$\begin{aligned}\pi_1(X) = \pi_1(X') &= \pi_1(U \cup V) \\ &= \pi_1(U) * \pi_1(V) \\ &= 0 * \mathbb{Z} \\ &= \mathbb{Z}.\end{aligned}$$

Therefore, $\pi_1(X) = \mathbb{Z}$.

□

ued...

① Let M and N be connected n-dimensional manifold, $n \geq 3$. Prove that the fundamental group of the connected sum $M \# N$ is isomorphic to $\pi_1(M) * \pi_1(N)$.

Pf: Since M and N are manifolds, they are open sets.

Let $i_M: M \to M\#N$ be the inclusion from M to $M\#N$ and let $i_N: N \to M\#N$ be the inclusion from N to $M\#N$.

The $i_M(M)$ and $i_N(N)$ are both open and path-connected.

Observe that $M\#N = i_M(M) \cup i_N(N)$.

By definition of connected sums, there is an open ball B_M removed from M, and an open ball B_N removed from N such that the boundaries of B_M and B_N are identified.

Therefore, $i_M(M) \cap i_N(N)$ is the boundary of an open ball of an n-dimensional manifold, and it is path-connected, and nonempty.

Thus, $i_M(M) \cap i_N(N)$ is homeo. to S^{n-1}.

Since $n \geq 3$, we have that $n-1 \geq 2 \Rightarrow \pi_1(S^{n-1}) = 0$

Since $i_M(M) \cap i_N(N)$ is simply-connected, we can use the following version of Van-Kampen:

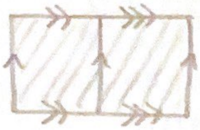
$$\begin{aligned}\pi_1(M\#N) &= \pi_1(i_M(M) \cup i_N(N)) \\ &= \pi_1(i_M(M)) * \pi_1(i_N(N)) \\ &= \pi_1(M) * \pi_1(N).\end{aligned}$$

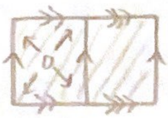
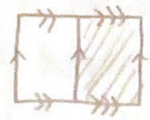
Therefore, $\pi_1(M\#N) = \pi_1(M) * \pi_1(N)$. $\square$

continued...

⑤ Compute the fundamental group of the space obtained from two tori by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other.

Pf: 

$X =$ 

Let $U =$  $\xrightarrow{\text{def. ret.}}$  $\xrightarrow{\text{def. ret.}}$  $\xrightarrow{\text{def. ret.}}$ 

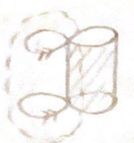
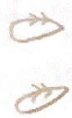
call this X'

U is open and path-connected.

we will use Van-Kampen on X'.

Let $U' =$  $\xrightarrow{\text{def. ret.}}$  $= \mathbb{T}^2$

U' is open, path-connected. $\pi_1(U') = \pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$.

Let $V' =$  $=$  $\xrightarrow{\text{def. ret.}}$  $\xrightarrow{\text{def. ret.}}$  $= S^1$

V' is open, path-connected. $\pi_1(V') = \pi_1(S^1) = \mathbb{Z}$.

Observe that $X' = U' \cup V'$.

$U' \cap V' =$  $\xrightarrow{\text{def. ret.}}$ 

$U' \cap V'$ is nonempty, path-connected. $\pi_1(U' \cap V') = 0$.

Since $U' \cap V'$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(X') = \pi_1(U' \cup V') = \pi_1(U') * \pi_1(V') = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$.

So $\pi_1(U) = \pi_1(X') = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$.

Let $V =$  open disk centered at missing point $\xrightarrow{\text{def. ret.}}$.

V is open and path-connected. $\pi_1(V) = 0$.

Observe that $X = U \cup V$.

$U \cap V =$  $\xrightarrow{\text{def. ret.}}$  $= S^1$

$U \cap V$ is nonempty and path-connected. $\pi_1(U \cap V) = \mathbb{Z}$

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Pf. We have two tori $S^1 \times S^1$ and are identifying $S^1 \times \{x_0\}$ in one torus with the corresponding $S^1 \times \{x_0\}$ in the other.

Identifying a point from one space to a point from another space is the wedge product, so we have that our new space is $S^1 \times (S^1 \vee S^1)$.

By Van-Kampen, we have that

$$\begin{aligned}\pi_1(S^1 \times (S^1 \vee S^1)) &= \pi_1(S^1) \times \pi_1(S^1 \vee S^1) \\ &= \mathbb{Z} \times (\pi_1(S^1) * \pi_1(S^1)) \\ &= \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z}).\end{aligned}$$

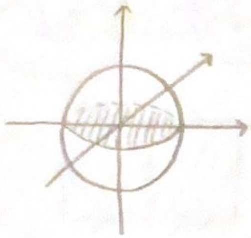
$\square$

continued...

⑥ Let $X = S^2 \cup \mathbb{D} \equiv \{(x,y,z) : x^2+y^2+z^2 = 1\} \cup \{(x,y,0) : x^2+y^2 \leq 1\}$.

Compute $\pi_1(X)$.

Pf: $S^2 \cup \mathbb{D} = X$



Let $U =$

$= S^2$

U is open and path-connected. $\pi_1(U) = \pi_1(S^2) = 0$.

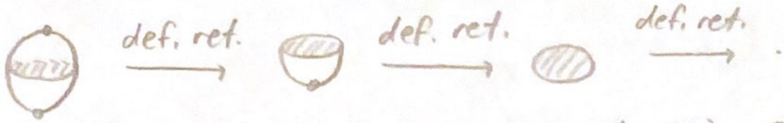
Let $V =$

$= S^2$

V is open and path-connected. $\pi_1(V) = \pi_1(S^2) = 0$.

Observe that $X = U \cup V$.

$U \cap V =$



$U \cap V$ is nonempty and path-connected. $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = 0 * 0 = 0$.

Therefore, $\pi_1(X)$ is trivial. □