Topology HW 6

① Let B be a closed disk in $\mathbb{R}^2$ and S be the boundary circle. Prove or disprove the following statements.

(i) Let $f: S \to B$ be a continuous map. Then there is a continuous map $F: B \to B$ such that $F|_S = f$.

Pf: True.

We can assume B is the unit disk centered at the origin.

The idea is that B is homeomorphic to $S \times [0,1]/(S \times \{0\})$, so we can get a map from $S \times [0,1] \to B$ by defining an appropriate homotopy.

Since B is contractible, the map f is null-homotopic.

There exists a homotopy $H: S \times [0,1] \to B$ such that $H(0,s) = x_0$, $H(1,s) = f(s)$ for all $s \in S$, where x_0 is some point of B.

Since H is constant on the set $S \times \{0\}$, H descends to a map $\bar{H}: (S \times [0,1])/(S \times \{0\}) \to B$.

For absolute clarity, let $g: B \to (S \times [0,1])/(S \times \{0\})$ be the homeomorphism defined by $g(w) = \begin{cases} (\|w\|, \frac{w}{\|w\|}), & w \neq 0 \\ (0,1), & w = 0. \end{cases}$

Then $\bar{H} \circ g: B \to B$ is the desired map.

It is continuous, and if $s \in S$, then $\bar{H} \circ g(s) = \bar{H}(1,s) = H(1,s) = f(s)$.

$\bar{H}$ is an extension of f to all of B. □

(ii) There is a continuous map $g: B \to S$ such that $g|_S$ is the identity map on S.

Pf: False.

If there exists a continuous map $g: B \to S$ s.t. $g|_S = Id_S$, then it's a retraction.

For retractions, $g_*: \pi_1(B) \to \pi_1(S)$ is surjective.

But $\pi_1(B)$ is trivial and $\pi_1(S) = \mathbb{Z}$ is infinite.

So g_* cannot be surjective. ↯

Therefore, no such g can exist. □

Continued...

② Let $\Sigma = \{(x,y,z) \in \mathbb{R}^3 ; z^2 = x^2+y^2-4\}$ with induced topology from $\mathbb{R}^3$.

(i) Find a universal covering space of Σ.

Pf: $z^2+4 = x^2+y^2$

Note that Σ is homeomorphic to a cylinder via the map $f: \Sigma \to S^1 \times \mathbb{R}$ given by $f(x,y,z) = \left(\frac{x}{\sqrt{z^2+4}}, \frac{y}{\sqrt{z^2+4}}, z\right)$.

It is clear that f is continuous, and f^{-1} is given by $g(x,y,z) = (\sqrt{z^2+4}\cdot x, \sqrt{z^2+4}\cdot y, z)$, which is also continuous.

We check that f actually carries Σ into $S^1 \times \mathbb{R}$: if $(x,y,z) \in \Sigma$, then $f(x,y,z) = \left(\frac{x}{\sqrt{z^2+4}}, \frac{y}{\sqrt{z^2+4}}, z\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right) \in S^1 \times \mathbb{R}$.

We also confirm that g maps $S^1 \times \mathbb{R}$ into Σ: if $(x,y,z) \in S^1 \times \mathbb{R}$, then $g(x,y,z) = (\sqrt{z^2+4}\cdot x, \sqrt{z^2+4}\cdot y, z)$, so that the sum of the squares of the first two coordinates is $(z^2+4)\cdot(x^2+y^2) = z^2+4$, as needed.

Since homeomorphic spaces have the same universal covering space, it suffices to determine the covering space of the cylinder.

Products of covering maps are covering maps, so it suffices to find the universal cover of S^1 and $\mathbb{R}$, both of which are $\mathbb{R}$.

So $\mathbb{R}^2$ is the universal covering space of the cylinder (because $\mathbb{R}^2$ is simply connected). □

(ii) Let $B = \Sigma/\sim$ be the quotient space where $\sim$ is the equivalence relation generated by the relation $(x,y,z) \sim (x,y,-z)$. Is the quotient map $q: \Sigma \to B$ a covering map? Prove your assertion.

Pf: No, q is not a covering map.

Notice that Σ is connected (Σ is homeo. to a cylinder).

If a covering map has a connected domain, then every fiber of the map has the same cardinality.

Hence, if q is covering, each fiber $q^{-1}(x)$ must have the same size.

Notice that $q^{-1}([(2,2,2)]) = \{(2,2,-2),(2,2,2)\}$ has size 2, and $q^{-1}([(2,0,0)]) = \{(2,0,0)\}$ has size 1.

This is a contradiction.

Thus, we conclude that q cannot be a covering map. □

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A topological space X is said to be compactly generated if the property below holds: Any subset A of X is closed in X if and only if $A \cap K$ is closed in K for every compact $K \subset X$.

(i) Show that a first countable space is compactly generated.

Pf: Let X be a first countable space.

- If $A \subseteq X$ is closed, then by definition of the subspace topology, $A \cap K$ is closed in K for every compact $K \subseteq X$.
- If $A \subseteq X$ is not closed, then we will show that there is a compact set $K \subseteq X$ such that $A \cap K$ is not closed in K.

Since A is not closed, there is an $x \in \bar{A} \setminus A$.

So there is a sequence of points $\{x_i\}_{i=1}^{\infty}$ in A converging to $x \in X$.

Let $K = \{x\} \cup \{x_i\}_{i=1}^{\infty}$.

We claim that K is compact.

Let $\{U_\alpha\}$ be an open cover of K. Then there is a U_{α_0} that contains x.

Since the sequence $\{x_i\}_{i=1}^{\infty}$ converges to x, by definition, there is an $N \in \mathbb{N}$ such that $x_i \in U_{\alpha_0}$ for all $i \geq N$.

Then since $\{U_\alpha\}$ is a cover of K, there exist $U_{\alpha_1}, \dots, U_{\alpha_{N-1}}$ such that $x_i \in U_{\alpha_i}$ for $1 \leq i \leq N-1$.

Therefore, $\{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_N}\}$ is a finite open cover of K, so K is compact.

However, $A \cap K = \{x_i\}_{i=1}^{\infty}$, so $A \cap K$ is not closed since it does not contain one of its limit points, namely x.

Thus, we conclude that X is compactly generated. $\square$

(ii) A map $f: X \to Y$ between two topological spaces is called proper if for every compact subset $K \subset Y$, the preimage $f^{-1}(K)$ is compact in X. Show that a continuous map $f: X \to Y$ from a compact space X to a Hausdorff space Y is proper.

Pf: Let $K \subseteq Y$ be compact.

Compact subsets of Hausdorff spaces are closed, so K must be closed.

Since f is continuous and K is closed, we have that $f^{-1}(K)$ is closed in X.

Closed subsets of compact spaces are compact.

Since $f^{-1}(K)$ is closed, X is compact, and $f^{-1}(K) \subseteq X$, we have that $f^{-1}(K)$ is compact.

Therefore, f is proper. $\square$