uary 2014

① Let $X, Y$ be topological spaces, $Y$ Hausdorff, and let $A \subset X$ be a nonempty set.

(a) Suppose that $f: A \to Y$ is continuous, where $A$ is equipped with the subspace topology. Prove that if there exists a continuous extension of $f$ to $\bar{A}$, it is unique.

Pf: Suppose there exists a continuous extension of $f$ to $\bar{A}$.

Assume that $g, h: \bar{A} \to Y$ are such continuous extensions s.t. $g \neq h$.

Since $g \neq h$, we know $\exists$ some $x \in \bar{A}$ s.t. $g(x) \neq h(x)$.

Let $U$ be an open nbhd of $g(x)$ and $V$ an open nbhd of $h(x)$ s.t. $U \cap V = \emptyset$ (we can do this because $Y$ is Hausdorff).

Since $g, h$ are continuous and $U, V$ are open in $Y$, we have that $g^{-1}(U), h^{-1}(V)$ are open in $\bar{A}$.

Let $S = \{x \in \bar{A} : g(x) \neq h(x)\}$. We WTS that $S$ is open.

We have that $x \in g^{-1}(U)$ and $x \in h^{-1}(V)$.

Let $W := g^{-1}(U) \cap h^{-1}(V)$.

$W$ is open because finite intersection of open sets is open, and $W$ is nonempty ($x \in W$, it is a nbhd of $x$).

Let $y \in W = g^{-1}(U) \cap h^{-1}(V)$.

We have that $g(y) \subseteq U$ and $h(y) \subseteq V \Rightarrow g(y) \neq h(y)\ \forall y \in W$ because $U \cap V = \emptyset$. Therefore, $y \in S$.

Thus, $x \in \underbrace{W}_{\text{open}} \subseteq S$, so $S$ is open.

$\Rightarrow \bar{A} \setminus S = \{x \in \bar{A} : g(x) = h(x)\}$ is closed.

$A \subseteq \bar{A} \setminus S$ and $\bar{A} \setminus S$ is closed $\Rightarrow \bar{A} \subseteq \bar{A} \setminus S$.

Therefore, $S$ must be empty since $\bar{A} = \bar{A} \setminus S$.

$S = \{x \in \bar{A} : g(x) = h(x)\} = \emptyset \Rightarrow g = h$ ↯

Thus, if there exists a continuous extension of $f$ to $\bar{A}$, it is unique. $\square$

Continued...

(b) Assume that $A$ is connected in the subspace topology. Prove that $\bar{A}$ is connected in the subspace topology.

Pf: Suppose that $A$ is connected.

Assume that $\bar{A}$ is not connected.

Then we can write $\bar{A} = U \cup V$, where $U, V$ are open, disjoint, and nonempty.

Since $A \subseteq \bar{A}$ and $A$ is connected, we have that $A \subseteq U$ or $A \subseteq V$.

WLOG, suppose $A \subseteq U$.

Then $V$ contains the limit points that are not in $A$.

Recall that $x$ is a limit point of $A$ if every deleted nbhd of $x$ intersects $A$.

Let $x \in \bar{A} \cap V$.

We want to find a deleted nbhd of $x$ that does not intersect $A$.

Let $V \setminus \{x\}$ be the deleted nbhd.

Then $V \setminus \{x\} \cap A = \emptyset$ because $V \setminus \{x\} \subseteq V$, $A \subseteq U$, and $U \cap V = \emptyset$.

Therefore, $x$ is not a limit point of $A$. ↯

Thus, $\bar{A}$ is connected.

□

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② Suppose that $X$ is a topological space homeomorphic to an open subset of a compact Hausdorff space. Prove that $X$ is locally compact (= every point has a nbhd contained in a compact set).

Pf: Let $U \subseteq Y$ be an open subset of $Y$, where $Y$ is a compact Hausdorff space.

Suppose $X$ is homeo. to $U$, and let $f: X \to U$ be a homeo.

Since $Y$ is compact and Hausdorff, we have that $Y$ is normal.

If $y \in U \subseteq Y$, then $Y \setminus U$ is a closed set that does not contain $y$.

Since $Y$ is normal ($\Rightarrow$ regular), we have $Y \setminus U \subseteq V$ open and $y \in W$ open s.t. $V \cap W = \emptyset$.

We want a compact set $K$ s.t. $W \subseteq K \subseteq U$.

Observe that $W$ is disjoint from $V$ and $Y \setminus U \subseteq V$.

$y \in W \subseteq Y$ $(W \cap V = \emptyset)$, so $y \in W \subseteq Y \setminus V$.

Since $V$ is open, $Y \setminus V$ is closed.

Since $Y$ is a compact Hausdorff space, a set is compact iff closed. So we have that $y \in W \subseteq \underbrace{Y \setminus V}_{\Rightarrow \text{ compact}}$ closed

Observe that $Y \setminus U \subseteq V \Rightarrow Y \setminus V \subseteq U$.

So we have that $y \in W \subseteq \underbrace{Y \setminus V}_{\text{compact}} \subseteq U$

Therefore, since $y \in U$ was arbitrary, we have shown that every point has a nbhd contained in a compact set.

Thus, $X$ is locally compact. $\square$

Continued...

⑤ Show that $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^2$.

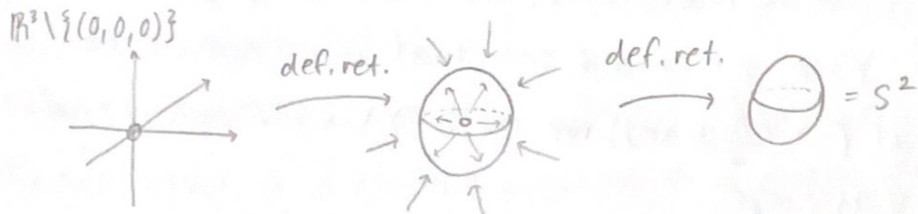
Pf: Assume that $\mathbb{R}^3$ is homeo. to $\mathbb{R}^2$.

Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be a homeomorphism.

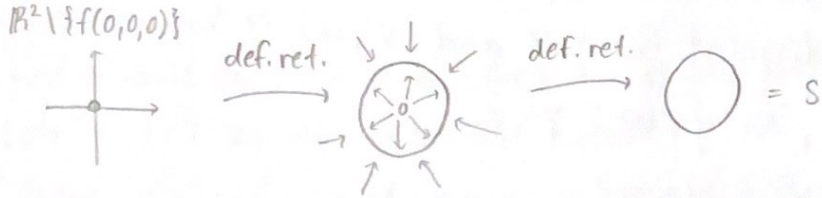
Remove a point from each space, then

$\tilde{f}: \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{f(0,0,0)\}$ is a homeomorphism.

However, observe that $\pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}) = \pi_1(S^2) = 0$,




and $\pi_1(\mathbb{R}^2 \setminus \{f(0,0,0)\}) = \pi_1(S^1) = \mathbb{Z}$.




So $\pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}) \neq \pi_1(\mathbb{R}^2 \setminus \{f(0,0,0)\})$

$0 \neq \mathbb{Z}$

Therefore, $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^2$.

$\square$

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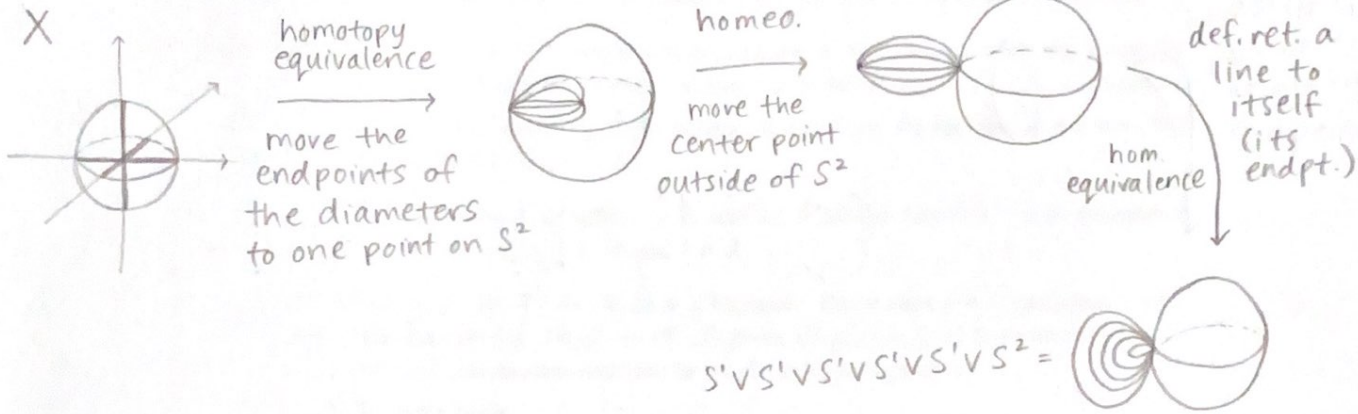
Let $X$ be the subspace of $\mathbb{R}^3$ equal to the union of the unit sphere with the three line segments $\{(0,0,z) : |z| \le 1\} \cup \{(0,y,0) : |y| \le 1\} \cup \{(x,0,0) : |x| \le 1\}$. Compute the fundamental group of $X$ based at $(1,0,0)$.

Pf: Since $X$ is path-connected, the fundamental group is independent of the base point (up to isomorphism).




$$\begin{aligned} \text{Therefore, } \pi_1(X) &= \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^2) \\ &= \underbrace{\pi_1(S^1) * \dots * \pi_1(S^1)}_{5 \text{ times}} * \pi_1(S^2) \\ &= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{5 \text{ times}} * 0 \\ &= \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \end{aligned}$$

Each space $(S^1, S^2)$ is locally Euclidean, so the wedge point has a nbhd in each space that def. ret. to the wedge point. So we can take the fund. gp. as follows.