...ary 2015

① Let $A$ and $B$ denote subsets of a topological space $X$. Prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Pf: First we will show $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Observe that $A \subseteq A \cup B \subseteq \overline{A \cup B}$, so $\overline{A \cup B}$ is a closed set containing $A$.

The smallest closed set containing $A$ is $\bar{A}$, so we have $\bar{A} \subseteq \overline{A \cup B}$.

Likewise, observe that $B \subseteq A \cup B \subseteq \overline{A \cup B}$, so $\overline{A \cup B}$ is a closed set containing $B$.

The smallest closed set containing $B$ is $\bar{B}$, so we have $\bar{B} \subseteq \overline{A \cup B}$.

Therefore, $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Now we will show $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

To do so, we will show that $x \notin \bar{A} \cup \bar{B}$, then $x \notin \overline{A \cup B}$.

Recall that if $x \in \bar{A}$, then every nbhd $U$ of $x$ is s.t. $U \cap A \neq \emptyset$.

If $x \notin \bar{A}$, then $\exists$ an open nbhd $U_1$ of $x$ s.t. $U_1 \cap A = \emptyset$.

If $x \notin \bar{B}$, then $\exists$ an open nbhd $U_2$ of $x$ s.t. $U_2 \cap B = \emptyset$.

Observe that $x \in U_1 \cap U_2$ and $U_1 \cap U_2$ is open (finite intersection of open sets is open) and nonempty ($x \in U_1 \cap U_2$).

Observe that $(U_1 \cap U_2) \cap (A \cup B) = \emptyset$.

Therefore, $x \notin \overline{A \cup B}$ since $\exists$ a nbhd $(U_1 \cap U_2)$ of $x$ s.t. $U_1 \cap U_2$ and $A \cup B$ are disjoint. Thus, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

We conclude that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

$\square$

continued...

(2) Let $f_1, f_2 : X \to Y$ be continuous maps from a topological space $X$ to a Hausdorff space $Y$. Show that the set of points $\{x \in X : f_1(x) = f_2(x)\}$ is a closed set.

Pf: Let $A = \{x \in X : f_1(x) = f_2(x)\}$.

Then $X \setminus A = \{x \in X : f_1(x) \neq f_2(x)\}$.

We will show that $X \setminus A$ is open ($\Rightarrow A$ is closed).

Let $x \in X \setminus A$. Then $f_1(x), f_2(x) \in Y$ s.t. $f_1(x) \neq f_2(x)$.

Since $Y$ is Hausdorff and $f_1(x) \neq f_2(x)$, there exist open nbhds $U$ of $f_1(x)$ and $V$ of $f_2(x)$ s.t. $U \cap V = \emptyset$.

Since $f_1, f_2$ are cts and $U, V$ are open, we have that $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in $X$.

Observe that $x \in f_1^{-1}(U)$ and $x \in f_2^{-1}(V)$.

Let $W := f_1^{-1}(U) \cap f_2^{-1}(V)$.

$W$ is open since it is the finite intersection of open sets.

$W$ is nonempty since $x \in f_1^{-1}(U) \cap f_2^{-1}(V) = W$.

We WTS that $W \cap A = \emptyset$.

Let $y \in W = f_1^{-1}(U) \cap f_2^{-1}(V)$

$$\left.\begin{array}{l} \text{If } y \in f_1^{-1}(U), \text{ then } f_1(y) \in U \\ \text{If } y \in f_2^{-1}(V), \text{ then } f_2(y) \in V \end{array}\right\} \text{ but } U \cap V = \emptyset, \text{ so } f_1(y) \neq f_2(y) \ \forall y \in W.$$

Therefore, $W \cap A = \emptyset$.

Thus, we have $x \in \overset{\text{open}}{W} \subseteq X \setminus A$, so $X \setminus A$ is open.

We conclude that $A = \{x \in X : f_1(x) = f_2(x)\}$ is closed.

$\square$

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(5) Let $X$ be a topological space, and let $A \subseteq X$ be a subset. Denote by $Int(A)$ and $\partial A$ the interior and boundary of $A$, respectively. Either prove the following statement, or give a counter-example.

1. If $A$ is connected, then $int(A)$ is connected.

<u>Pf:</u> This is false.

Let $A = \underbrace{\{(x,y) \in \mathbb{R}^2 : x<0\}}_{\text{left half of } \mathbb{R}^2} \cup \underbrace{\{(x,y) \in \mathbb{R}^2 : x>0\}}_{\text{right half of } \mathbb{R}^2} \cup \{(0,0)\}$

Observe that $A$ is connected.

We have that $Int(A) = \{(x,y) \in \mathbb{R}^2 : x<0\} \cup \{(x,y) \in \mathbb{R}^2 : x>0\}$, which is a separation.

Observe that $Int(A)$ is disconnected (y-axis is missing from $\mathbb{R}^2$).

There is no open ball $B(r,(0,0))$ around $(0,0)$ where $r>0$ s.t. $B(r,(0,0)) \subseteq A$.

Therefore, $(0,0) \notin Int(A)$. $\square$

2. If both $Int(A)$ and $\partial A$ are connected, then $A$ is connected.

<u>Pf:</u> This is false.

Consider $\mathbb{R}$ with the standard topology.

The connected sets are $\emptyset$, $\{x\}$, and intervals.

The $Int(A)$ can be either $\emptyset$ or $(a,b)$ ($a,b$ could be $\infty$).

If $Int(A) = (a,b)$, then $a,b \in \partial A$, but $(a,b) \notin \partial A \Rightarrow \partial A$ is disconnected.

Therefore, $Int(A) = \emptyset$.

Recall that $\partial A = \{x \in \mathbb{R} : \text{every interval containing } x \text{ intersects } A \text{ and } \mathbb{R} \setminus A\}$.

Every $x \in \mathbb{R}$ satisfies $B(x,\varepsilon) \cap A \neq \emptyset$ and $B(x,\varepsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$.

If $A$ intersects every nonempty open subset of $X$, then $A$ is dense.

Let $A = \mathbb{Q}$. $A$ is a dense subset of $\mathbb{R}$ and $\mathbb{R} \setminus A$ is dense.

$Int(A) = \emptyset \Rightarrow Int(A)$ is connected

$\partial A = \mathbb{R} \Rightarrow \partial A$ is connected

But $A = \mathbb{Q} = ((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cup ((\sqrt{2}, \infty) \cap \mathbb{Q}) \Rightarrow A$ is disconnected.

open, nonempty, disjoint. $\square$

continued...

(4) Let $q: E \to X$ be a covering map with $q^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $E$ is compact if and only if $X$ is compact.

Pf: • Suppose $E$ is compact.
Since $q$ is surjective, we have $q(E) = X$.
The cts image of a cpt space is cpt.
Since $q$ is cts, $E$ is cpt, we have that $q(E) = X$ is compact.

• Suppose $X$ is compact.
Let $\mathcal{U} = \{\tilde{U}_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of $E$.
Claim: Each point $x \in X$ has a nbhd $V_x$ s.t. $q^{-1}(x)$ can be covered by finitely many elements of $\mathcal{U}$: $q^{-1}(V_x) \subseteq \bigcup_{i=1}^{n_x} \tilde{U}_i^x$.
Assuming the claim is true, notice that $X = \bigcup_{x \in X} V_x = \bigcup_{i=1}^{m} V_{x_i}$ b/c $X$ is compact.
$E = q^{-1}(X) = \bigcup_{i=1}^{m} q^{-1}(V_x) \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_{x_i}} \tilde{U}_j^{x_i}$ where $\tilde{U}_j^{x_i} \in \mathcal{U}$.
Let $N = \max(n_{x_1}, n_{x_2}, \dots, n_{x_m})$.
$E$ is covered by $m \cdot N$ open sets from $\mathcal{U}$.

□

Now we prove the claim. Let $x \in X$.
First, let $V$ be any evenly covered nbhd of $x$. Since $q^{-1}(x)$ is finite, let $q^{-1}(x) = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$. We have $q^{-1}(V) = \bigcup_{i=1}^{n} \tilde{V}_i$ because exactly one point in each $\tilde{V}_i$ gets mapped to $x$ (b/c $q$ is homeo. $q: \tilde{V}_i \to V$).
We can assume $\tilde{x}_i \in \tilde{V}_i$ (otherwise relabel)
Let $\tilde{U}_i$ be any element of $\mathcal{U}$ s.t. $\tilde{x}_i \in \tilde{U}_i$. Let $\tilde{W}_i = \tilde{U}_i \cap \tilde{V}_i$, open and $\tilde{x}_i \in \tilde{W}_i$.
observe that: (1) $q(\tilde{W}_i) \subseteq q(\tilde{V}_i) = V \;\; \forall i = 1, \dots, n$
(2) If $y \in V$, then $q^{-1}(y)$ contains exactly $n$ points
(3) $q(\tilde{x}_i) = x \in q(\tilde{W}_i) \;\; \forall i$

$V_x = \bigcap_{i=1}^{n} q(\tilde{W}_i)$ open b/c fin. intersection of open sets (covering maps are open)
By (3), $x \in V_x$. Finally, we show $q^{-1}(V_x) \subseteq \bigcup_{i=1}^{n} \tilde{U}_i$. This will finish proof of claim.
It suffices to show $q^{-1}(y) \subseteq \bigcup_{i=1}^{n} \tilde{U}_i$ for each $y \in V_x$.
By (1), $V_x \subseteq q(\tilde{W}_i) \subseteq V$, so $q^{-1}(y)$ has exactly $n$ points (using (2)).
$y \in V_x = \bigcap_{i=1}^{n} q(\tilde{W}_i)$, so $\exists\, \tilde{y}_i \in \tilde{W}_i$ s.t. $q(\tilde{y}_i) = y$. We know $\tilde{W}_i$ are disjoint b/c $\tilde{W}_i \subseteq \tilde{V}_i$ and slices are pairwise disjoint, so $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n\} \subseteq q^{-1}(y)$.
$\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n\} = q^{-1}(y)$ b/c both sides are $n$-element sets.
$\tilde{y}_i \in \tilde{W}_i \subseteq \tilde{U}_i$, so $q^{-1}(y) \subseteq \bigcup_{i=1}^{n} \tilde{U}_i$.

□

...ued...

(5) Let $\mathbb{P}^2$ denote the (real) projective plane. Prove that any continuous map $f: \mathbb{P}^2 \to S^1$ is null-homotopic, i.e., homotopic to a constant map.

Pf: We will use the general lifting lemma to show $\exists$ a lift $\tilde{f}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{P}^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Observe that $\mathbb{P}^2$ is path-connected because it is the continuous image of $S^2$ which is path-conn. (and the cts image of path-conn. is path-conn.)

Observe that $\mathbb{P}^2$ is locally path-connected because $q: S^2 \to \mathbb{P}^2$ is a local homeomorphism.

Let $p: \mathbb{R} \to S^1$ be a covering map (the exp. map).

In order to use the general lifting lemma, it remains to show that $f_*(\pi_1(\mathbb{P}^2)) \subseteq p_*(\pi_1(\mathbb{R}))$.

Observe that $\pi_1(\mathbb{R}) = 0$, and $\pi_1(\mathbb{P}^2) = \overset{\text{finite}}{\mathbb{Z}/2\mathbb{Z}}$ so $f_*(\pi_1(\mathbb{P}^2))$ is finite.

We have that $f_*(\pi_1(\mathbb{P}^2)) \subseteq \pi_1(S^1) = \mathbb{Z}$.

The only finite subgp of $\mathbb{Z}$ is $0$, so $f_*(\pi_1(\mathbb{P}^2)) = 0 \subseteq p_*(\pi_1(\mathbb{R}))$ ✓

Therefore, by the general lifting lemma, there exists a lift $\tilde{f}: \mathbb{P}^2 \to \mathbb{R}$ (unique).

Any continuous map into a contractible space is null-homotopic.

Since $\tilde{f}$ is cts and $\mathbb{R}$ is contractible, we have that $\tilde{f}$ is null-homotopic.

If $\tilde{f}$ is null-homotopic, then $f$ is null-homotopic (if $H$ is the homotopy between $\tilde{f}$ and a constant, then $p \circ H$ is the homotopy between $f$ and a constant).

Therefore, since $\tilde{f}$ is null-homotopic, $f$ is null-homotopic.

Thus, any continuous map $f: \mathbb{P}^2 \to S^1$ is null-homotopic.

□

continued...

⑥ Let $n \geq 3$ be an integer. Suppose $M$ and $N$ are connected $n$-dimensional manifolds. Prove that the fundamental group of the connected sum $M \# N$ is isomorphic to $\pi_1(M) * \pi_1(N)$.

Pf: Since $M$ and $N$ are manifolds, they are open sets.

Let $i_M : M \to M\#N$ be the inclusion map from $M$ to $M\#N$ and let $i_N : N \to M\#N$ be the inclusion map from $N$ to $M\#N$.

Then $i_M(M)$ and $i_N(N)$ are both open and path-connected.

Observe that $i_M(M) \cup i_N(N) = M\#N$.

By definition of connected sums, there is an open ball $B_M$ removed from $M$ and an open ball $B_N$ removed from $N$ such that the boundaries of $B_M$ and $B_N$ are identified.

Therefore, $i_M(M) \cap i_N(N)$ is the boundary of an open ball of an $n$-dim. manifold, and it is path-conn.. Thus, $i_M(M) \cap i_N(N)$ is homeo. to $S^{n-1}$.

Since $n \geq 3$, we have that $n-1 \geq 2 \Rightarrow \pi_1(S^{n-1}) = 0$.

Since $i_M(M) \cap i_N(N)$ is simply connected, we can use the following version of Van-Kampen,
$$\begin{aligned}\pi_1(M\#N) &= \pi_1(i_M(M) \cup i_N(N)) \\ &= \pi_1(i_M(M)) * \pi_1(i_N(N)) \\ &= \pi_1(M) * \pi_1(N).\end{aligned}$$

□