uary 2016

① Let X be the set of all points $(x,y) \in \mathbb{R}^2$ such that $y=1$ or $y=-1$. Let M be the quotient of X by the equivalence relation generated by $(x,-1) \sim (x,1)$ for all $x \neq 0$. Show that M is not Hausdorff.

Pf: $X/\sim = M$

Let $q: X \to X/\sim = M$ be the quotient map.

Recall that since q is a quotient map,

$U \subseteq X/\sim = M$ is open iff $q^{-1}(U) \subseteq X$ is open.

Consider the points $(0,-1)$ and $(0,1)$.

We WTS that every nbhd of $(0,-1)$ intersects every nbhd of $(0,1)$.

Let $U \subseteq M$ be an open nbhd of $(0,-1)$. Then $q^{-1}(U)$ is open in X.

We know that $(0,-1) \in q^{-1}(U)$ and since $q^{-1}(U)$ is open

$(0,-1) \in \{(x,-1) : |x| < a\} \subseteq q^{-1}(U)$ ($q^{-1}(U)$ is saturated)

and $\{(x,1) : 0 < |x| < a\} \subseteq q^{-1}(U)$.

Let $V \subseteq M$ be an open nbhd of $(0,1)$. Then $q^{-1}(V)$ is open in X.

We know that $(0,1) \in q^{-1}(V)$ and since $q^{-1}(V)$ is open

$(0,1) \in \{(x,1) : |x| < b\} \subseteq q^{-1}(V)$ ($q^{-1}(V)$ is saturated)

and $\{(x,-1) : 0 < |x| < b\} \subseteq q^{-1}(V)$

Let $0 < c < \min\{a,b\}$.

Then $(c,1) \in q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V)$.

If $U \cap V = \emptyset$, then $q^{-1}(U \cap V) = \emptyset$.

Since $(c,1) \in q^{-1}(U \cap V) \neq \emptyset$, we have that $U \cap V \neq \emptyset$.

Therefore, we have shown that every nbhd of $(0,-1)$ intersects every nbhd of $(0,1)$.

Thus, M is not Hausdorff.

$\square$

continued...

② Suppose $f: X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff. Prove that f is a homeomorphism.

Pf: Since f is a continuous bijection, it suffices to show that f is closed.

Let $K \subseteq X$ be a closed subset.

Closed subsets of compact spaces are compact.

Since K is closed and X is compact, we have that K is compact.

The cts image of a compact set is compact.

Since f is cts and K is cpt, we have that $f(K) \subseteq Y$ is compact.

Compact subsets of Hausdorff spaces are closed.

Since $f(K)$ is cpt and Y is Hausdorff, we have that $f(K)$ is closed.

Therefore, if $K \subseteq X$ is closed, then $f(K) \subseteq Y$ is closed.

Thus, f is closed.

We conclude that f is a homeomorphism.

$\square$

ued...

) Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to \mathbb{T}^2$ is null-homotopic.

Pf: We would like to use the general lifting lemma to show that the lift $\tilde{f}: X \to \mathbb{R}^2$ exists.

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{P}^2 \end{tikzcd}$$

Observe that X is path-conn. and locally path-conn.
Let $p: \mathbb{R}^2 \to \mathbb{T}^2$ be a covering map, where p is the product of two exp. maps
$\left(\begin{array}{l} \text{let } p_1: \mathbb{R} \to S^1 \text{ exp.} \\ p_2: \mathbb{R} \to S^1 \text{ exp.} \end{array} \right\} p := p_1 \times p_2 : \begin{array}{l} \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \\ \mathbb{R}^2 \to \mathbb{T}^2 \end{array} \Big)$

We WTS $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}^2))$.
Observe that $\pi_1(\mathbb{R}^2) = 0$ because $\mathbb{R}^2$ is convex.
We WTS $f_*(\pi_1(X)) = 0$.
Since $\pi_1(X)$ is finite, $f_*(\pi_1(X))$ is finite.
Observe that $f_*(\pi_1(X)) \subseteq \pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.
The only finite subgp. of $\mathbb{Z} \times \mathbb{Z}$ is 0.
Therefore, $f_*(\pi_1(X)) = 0 \subseteq p_*(\pi_1(\mathbb{R}^2))$ ✓
Thus, by the general lifting lemma, the lift $\tilde{f}: X \to \mathbb{R}^2$ exists.
Observe that $\mathbb{R}^2$ convex $\Rightarrow$ $\mathbb{R}^2$ is contractible.
Any cts fn. into a contractible space is null-homotopic.
Therefore, $\tilde{f}: X \to \mathbb{R}^2$ is null-homotopic.
If $\tilde{f}$ is null-homotopic, then so is f.
(If H is a homotopy btwn $\tilde{f}$ and a constant, then $p \circ H$ is a homotopy btwn f and a constant)
Thus, f is null-homotopic, as desired.

□

continued..

(4) Let A be a subset of a topological space X. Suppose that $r: X \to A$ is a retraction of X onto A, i.e., r is a continuous map such that the restriction of r to A is the identity map of A.

(1) Show that if X is Hausdorff, then A is a closed subset.

Pf: To show that A is closed, we will show that $X \setminus A$ is open:
for $x \in X \setminus A$, $\exists\, U$ open s.t. $x \in U \subseteq X \setminus A$.
Let $x \in X \setminus A$. Then $r(x) \in A$.
Let U, V be the disjoint open nbhds of $x, r(x)$, respectively.
(Such U, V exist because X is Hausdorff)
Then $U \cap V = \emptyset$. $U \subseteq X$, $V \subseteq A$.
Since V is open and r is cts, $r^{-1}(V)$ is open in X.
We have that $x \in r^{-1}(V)$ and $x \in U$, so $x \in r^{-1}(V) \cap U$.
Therefore, $r^{-1}(V) \cap U$ is nonempty and open (finite intersection of open sets is open)
If $(r^{-1}(V) \cap U) \cap A$, then assume $a \in r^{-1}(V) \cap U$ and $a \in A$.

$$\left.\begin{array}{l} \text{Since } a \in A, \text{ we have that } r(a) = a \in V \\ \text{Since } a \in r^{-1}(V) \cap U, \text{ we have that } a \in U \end{array}\right\} \Rightarrow a \in U \cap V = \emptyset \quad \lightning$$

Therefore, $(r^{-1}(V) \cap U) \cap A = \emptyset$.
Thus, $r^{-1}(V) \cap U$ is an open set s.t. $x \in r^{-1}(V) \cap U \subseteq X \setminus A$.
We conclude that $X \setminus A$ is open $\Rightarrow$ A is closed. $\square$

(2) Let $a \in A$. Show that $r_*: \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Pf: Recall that $r_*([\alpha]) = [r \circ \alpha]$.
Let $[\beta] \in \pi_1(A, a)$.
We WTS $\exists\, [\alpha] \in \pi_1(X, a)$ s.t. $r_*([\alpha]) = [r \circ \alpha] = [\beta]$.
Observe that $\alpha(t) = \beta(t)\ \forall t$.
Since $A \subseteq X$, β can be thought of as a loop in X.
$r_*([\beta]) = [r \circ \beta] = [\beta]$
Therefore, $r_*: \pi_1(X, a) \to \pi_1(A, a)$ is surjective. $\square$

...ued...

Let S^n be an n-dimensional sphere in $\mathbb{R}^{n+1}$ centered at the origin. Suppose $f, g: S^n \to S^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Prove that f and g are homotopic.

Pf. Define $H: [0,1] \times S^1 \to S^1$ given by $H(t,x) = \dfrac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$.

Observe that H is the product and sum of continuous functions, and is therefore continuous.

First, we will show that H is well-defined (i.e., $(1-t)f(x) + tg(x) \neq 0$):

$$(1-t)f(x) + tg(x) = 0$$
$$(1-t)f(x) = -tg(x)$$
$$\|(1-t)f(x)\| = \|-tg(x)\| \quad \text{because } \|f(x)\| = 1, \ \|g(x)\| = 1$$
$$|(1-t)| = |-t|$$
$$1-t = t$$
$$1 = 2t$$
$$\Rightarrow t = \tfrac{1}{2}$$

$$(1-\tfrac{1}{2})f(x) = -\tfrac{1}{2}g(x)$$
$$\tfrac{1}{2}f(x) = -\tfrac{1}{2}g(x)$$
$\Rightarrow f(x) = -g(x)$, but we have that $f(x) \neq -g(x)$ for any $x \in S^n$.

Therefore, we conclude that H is well-defined.

Now, we will show that H is a homotopy:

$$H(0,x) = \frac{(1-0)f(x) + 0 \cdot g(x)}{\|(1-0)f(x) + 0\,g(x)\|} = \frac{f(x)}{\|f(x)\|} = f(x)$$

$$H(1,x) = \frac{(1-1)f(x) + 1 \cdot g(x)}{\|(1-1)f(x) + 1 \cdot g(x)\|} = \frac{g(x)}{\|g(x)\|} = g(x)$$

Therefore, H is a homotopy between f and g.

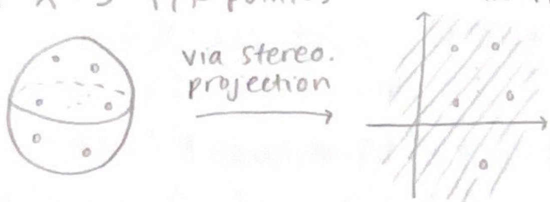
Thus, f and g are homotopic. $\square$

continued...

⑥ Let $k \geq 1$ be an integer. Compute the fundamental groups of the following spaces.

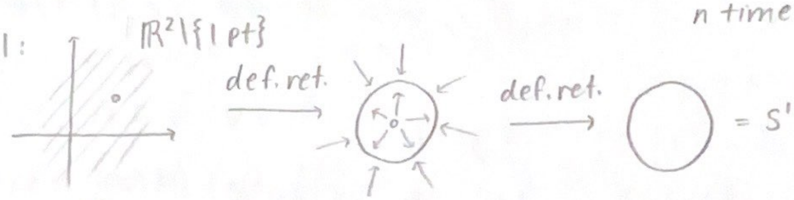
(1) The sphere S^2 with k points removed.

Pf: Let $X = S^2 \setminus \{k \text{ points}\}$ $\qquad \mathbb{R}^2 \setminus \{k-1 \text{ points}\}$



We will use induction to prove that $\pi_1(\mathbb{R}^2 \setminus \{n \text{ pts}\}) = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n \text{ times}}$

Base case: $n=1$:

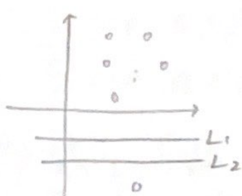


So $\pi_1(\mathbb{R}^2 \setminus \{1 \text{ pt}\}) = \pi_1(S^1) = \mathbb{Z}$

Assume that $\pi_1(\mathbb{R}^2 \setminus \{n-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n-1 \text{ times}}$ (Inductive step)

We WTS that this holds for $\pi_1(\mathbb{R}^2 \setminus \{n \text{ pts}\})$.

Let $\mathbb{R}^2 \setminus \{n \text{ pts}\}$ be drawn below. Since there is space between the removed points, we can draw two parallel lines between one point and the remaining $n-1$ points.



(U, V are open and path-connected)

Let U = everything below L_1 and V = everything above L_2.

Then by the base case $\pi_1(U) = \mathbb{Z}$ and by the inductive step $\pi_1(V) = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n-1 \text{ times}}$.

$U \cup V = \mathbb{R}^2 \setminus \{n \text{ pts}\}$

Observe that $U \cap V$ = the open strip of $\mathbb{R}^2$ between L_1 and L_2.

Since $U \cap V$ is convex, $\pi_1(U \cap V) = 0$.

Therefore, since $U \cap V$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(\mathbb{R}^2 \setminus \{n \text{ pts}\}) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V)$

$$= \mathbb{Z} * \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n-1 \text{ times}}$$

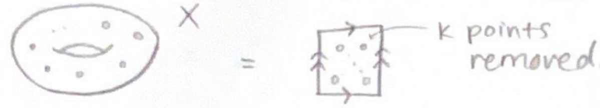
Therefore, $\pi_1(S^2 \setminus \{k \text{ pts}\}) = \pi_1(\mathbb{R}^2 \setminus \{k-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k-1 \text{ times}}$.

□

...ued...

) The torus $\mathbb{T}^2$ with k points removed.

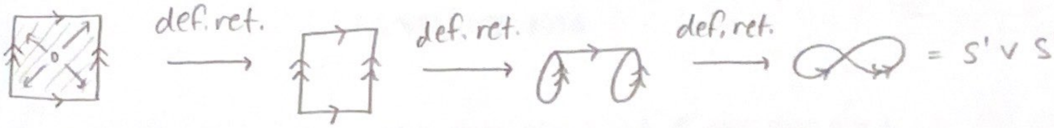
Pf:



Let $X = \mathbb{T}^2 \setminus \{k \text{ pts}\}$

We will use induction to show that $\pi_1(\mathbb{T}^2 \setminus \{n \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n+1 \text{ times}}$

Base case: $n=1$

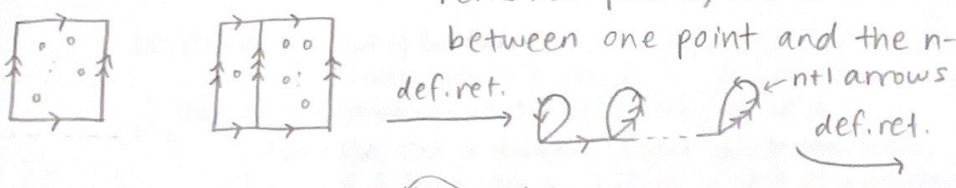


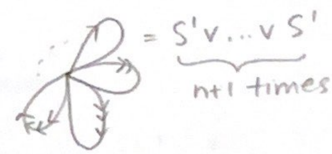
$\pi_1(\mathbb{T}^2 \setminus \{1 \text{ pt}\}) = \pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$.

Assume that $\pi_1(\mathbb{T}^2 \setminus \{n-1 \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n\text{-times}}$ (Inductive step)

We WTS $\pi_1(\mathbb{T}^2 \setminus \{n \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n+1 \text{ times}}$

Let $\mathbb{T}^2 \setminus \{n \text{ pts}\}$ be drawn below. Since there is space between the removed points, we can fix a parallel line between one point and the $n-1$ remaining points





$$\pi_1(\mathbb{T}^2 \setminus \{n \text{ pts}\}) = \pi_1(\overbrace{S^1 \vee \dots \vee S^1}^{n+1 \text{ times}})$$
$$= \underbrace{\pi_1(S^1) * \dots * \pi_1(S^1)}_{n+1 \text{ times}}$$
$$= \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n+1 \text{ times}}$$

Therefore, $\pi_1(\mathbb{T}^2 \setminus \{k \text{ pts}\}) = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k+1 \text{ times}}$.

$\square$