uary 2017:

① On $\mathbb{R}$ we consider the topology τ generated by the basis of all sets of the form (a,b) and all sets of the form $(a,b)\setminus K$, where $K := \{\frac{1}{n} : n \in \mathbb{N}\}$.

(a) Prove that $[0,1]$ is not compact in $(\mathbb{R},\tau)$.

Pf. We want an open cover of $[0,1]$ that does not admit a finite subcover.
Consider the sets $U_n = \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cup ((-1,2)\setminus K)$ for $n \geq 2$ and $U_1 = (\frac{1}{2}, 2) \cup ((-1,2)\setminus K)$. It is clear that the U_n are open.
We see that $\bigcup_{n=1}^{\infty} U_n = (-1,2)$ and $\frac{1}{n} \in U_m$ iff $n=m$.
To see this last point, notice that we can write $U_n = \{\frac{1}{n}\} \cup ((-1,2)\setminus K)$.
Hence the U_n form an open cover of $[0,1]$, but removing any U_n from the list would remove $\frac{1}{n}$ from their union.
Therefore, the open cover $\{U_n\}$ does not admit a finite subcover, so $[0,1]$ is not compact in $(\mathbb{R},\tau)$. $\square$

(b) Prove that $(\mathbb{R},\tau)$ is connected, but not path-connected.

continued.

② Let X be a set and τ and σ topologies on X so that τ is strictly finer (larger) than σ. Prove the following statements:

(a) If (X,τ) is compact and Hausdorff, then (X,σ) is not Hausdorff.

(b) If (X,σ) is compact and Hausdorff, then (X,τ) is not compact.

Hint: Consider the identity map on X.

- Recall that τ strictly finer than $\sigma \Rightarrow (X,\sigma) \subseteq (X,\tau)$.

Pf of (a): Consider the identity map on X: $id_X: (X,\tau) \to (X,\sigma)$.

Since $\sigma \subseteq \tau$, we have that for every $U \subseteq (X,\sigma)$ open, that $id_X^{-1}(U) = U \subseteq (X,\tau)$ is open.

Therefore, id_X is continuous.

Suppose (X,τ) is compact and Hausdorff.

Assume that (X,σ) is Hausdorff.

Let $K \subseteq (X,\tau)$ be a closed set.

Closed subsets of compact spaces are compact, so K is compact.

The cts image of a cpt set is cpt, so $id_X(K) = K$ is cpt in (X,σ).

Compact subsets of Hausdorff spaces are closed, so $id_X(K) = K$ is closed in (X,σ).

Therefore, id_X is a closed map.

Let K be a subset of (X,τ) s.t. K is closed in (X,τ), but not in (X,σ). Such a K exists because τ is strictly finer than σ.

Then $id_X(K)$ must be closed in (X,σ) since id_X is closed.

But id_X is the identity map, so $id_X(K) = K$, which is not closed in (X,σ). ↯ Contradiction since K is not closed in (X,σ).

Therefore, if (X,τ) is compact and Hausdorff, then (X,σ) is not Hausdorff. $\square$

...nued...

Pf of (b): Consider the identity map from part (a): $id_X : (X,\tau) \to (X,\sigma)$.

For the same reason as in part (a), id_X is continuous.

Suppose (X,σ) is compact and Hausdorff.

Assume (X,τ) is compact.

Let $K \subseteq (X,\tau)$ be a closed set.

Closed subsets of compact spaces are compact, so K is compact.

The cts image of cpt is cpt, so $id_X(K) = K$ is cpt in (X,σ).

Compact subsets of Hausdorff spaces are closed, so $id_X(K) = K$ is closed in (X,σ).

Therefore, id_X is a closed map.

Let K be closed in (X,τ), but not in (X,σ).

Closed subsets of compact spaces are compact, so we know that K is compact in (X,τ). In particular, K is compact in (X,τ), not in (X,σ).

The cts image of cpt is cpt, so $id_X(K) = K$ is compact in (X,σ) $\lightning$ Contradiction since we said that K is not compact (or closed) in (X,σ).

Therefore, if (X,σ) is compact and Hausdorff, then (X,τ) is not compact.

$\square$

continued..

③ Define $A = \{x \in \mathbb{R}^2 : \text{both coordinates of } x \text{ are rational}\}$.

$B = \{x \in \mathbb{R}^2 : \text{at least one coordinate of } x \text{ is rational}\}$.

Show that $\mathbb{R}^2 \setminus A$ is connected and $\mathbb{R}^2 \setminus B$ is not connected.

Pf: First we will show that $\mathbb{R}^2 \setminus A$ is connected.

Let $(x,y), (w,z) \in \mathbb{R}^2 \setminus A$.

Then either x or y is irrational and either w or z is irrational.

WLOG, suppose x is irrational.

We WTS $\exists$ a path from (x,y) to (w,z) that avoids A.

Since x is irrational, there exists a straight line path from (x,y) to (x,u) where u is irrational.

If w is irrational, then there exists a straight line path from (x,u) to (w,u) and then a straight line path from (w,u) to (w,z).

If z is irrational, then there exists a straight line path from (x,u) to (x,z) and then a straight line path from (x,z) to (w,z).

Therefore, we can construct a path between any two points in $\mathbb{R}^2 \setminus A$. Thus, $\mathbb{R}^2 \setminus A$ is path-connected $\Rightarrow$ connected.

Now we will show that $\mathbb{R}^2 \setminus B$ is not connected.

Observe that

$$\mathbb{R}^2 \setminus B = \{(x,y) \in \mathbb{R}^2 \setminus B : x < 0\} \cup \{(x,y) \in \mathbb{R}^2 \setminus B : x > 0\}$$

(we can do this b/c the points in $\mathbb{R}^2 \setminus B$ are of the form (irrational, irrational) and 0 is not irrational.)

Notice that the two sets are nonempty, open, and disjoint.

Therefore, this is a separation of $\mathbb{R}^2 \setminus B$.

Thus, $\mathbb{R}^2 \setminus B$ is not connected.

$\square$

* For (a), suffices to show path from (x,y) to (π, π).

...nued.

(4) Let X be a topological space and $f, g: X \to S^2$ two continuous maps. Show that if for every $x \in X$ the points $f(x)$ and $g(x)$ on S^2 are not antipodal to each other, then f and g are homotopic.

Pf: Suppose that for every $x \in X$ the points $f(x)$ and $g(x)$ on S^2 are not antipodal to each other, meaning $f(x) \neq -g(x)$.

Let $H: [0,1] \times X \to S^2$ be defined by $H(t,x) = \dfrac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$.

Observe that H is continuous because we are taking the sum and product of continuous functions.

First we will show that H is well-defined (i.e., $(1-t)f(x) + tg(x) \neq 0$):

$$(1-t)f(x) + tg(x) = 0 \Rightarrow (1-t)f(x) = -tg(x)$$

$$\|(1-t)f(x)\| = \|-tg(x)\|$$ because $\|f(x)\| = 1$, $\|g(x)\| = 1$

$$\|(1-t)\| = \|-t\|$$

$$1-t = t$$

$$1 = 2t$$

$$\Rightarrow t = \tfrac{1}{2}$$

So $(1-\frac{1}{2}) f(x) = -\frac{1}{2} g(x) \Rightarrow \frac{1}{2} f(x) = -\frac{1}{2} g(x)$

$$\Rightarrow f(x) = -g(x) \quad \lightning$$

This cannot happen b/c $f(x)$ and $g(x)$ are not antipodal to each other.

Therefore, $(1-t)f(x) + tg(x) \neq 0$, so H is well-defined.

Now we will show that H is a homotopy:

$$H(0,x) = \frac{(1-0)f(x) + 0 \cdot g(x)}{\|(1-0)f(x) + 0 \cdot g(x)\|} = \frac{f(x)}{\|f(x)\|} = f(x)$$

$$H(1,x) = \frac{(1-1)f(x) + 1 \cdot g(x)}{\|(1-1)f(x) + 1 \cdot g(x)\|} = \frac{g(x)}{\|g(x)\|} = g(x).$$

Therefore, H is a homotopy.

Thus, f and g are homotopic.

$\square$

Continued...

⑤ Compute the fundamental groups of the following spaces:

(a) The torus $\mathbb{T}^2 = S^1 \times S^1$.

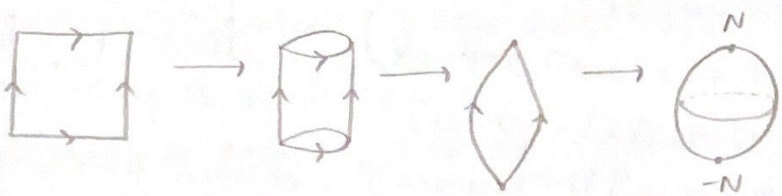
Pf. $\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

The fundamental group of a torus is $\mathbb{Z} \times \mathbb{Z}$. □

(b) The pinched torus $S^1 \times S^1 / S^1 \times \{1\}$.

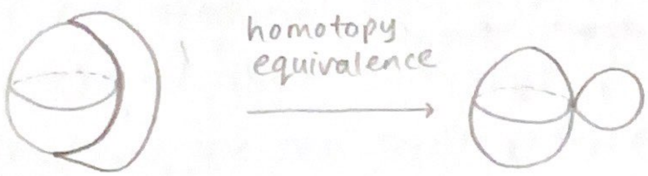
Pf. Let $X = S^1 \times S^1 / S^1 \times \{1\}$.

Here is a construction of the pinched torus:



This is S^2 with two points (N and $-N$) identified together.

X is homeo. the shere with two points identified.



$= S^2 \vee S^1$.

So $\pi_1(X) = \pi_1(S^1 \times S^1 / S^1 \times \{1\}) = \pi_1(S^2 \vee S^1)$

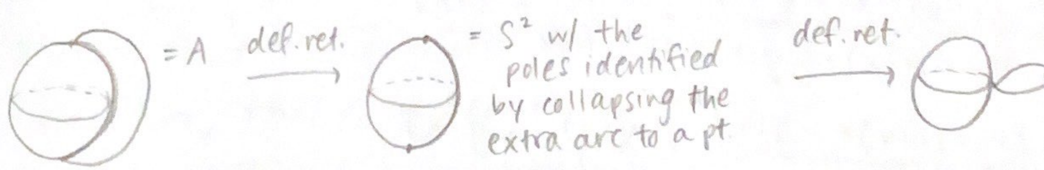
$= \pi_1(S^2) * \pi_1(S^1)$

$= 0 * \mathbb{Z}$

$= \mathbb{Z}$.

The fundamental group of the pinched torus is $\mathbb{Z}$. □

Let



$= S^2 \vee S^1$ by sliding the two endpoints of the arc to the same pt.

...ued.

(6) Assume $p: \tilde{X} \to X$ is a covering and both X and $\tilde{X}$ are path-connected. Assume A is a path-connected subset of X so that $i_*: \pi_1(A, a) \to \pi_1(X, a)$ is onto, for some $a \in A$, where $i: A \to X$ is the inclusion map. Prove that $p^{-1}(A)$ is path-connected as well.

Pf: Let $x, y \in p^{-1}(A)$ be arbitrary points.
Since $p^{-1}(A)$ is the complete preimage of a subspace of X under p, we know that the restriction $p: p^{-1}(A) \to A$ is a covering map.
Let $\tilde{A} := p^{-1}(A)$.

· Since $\tilde{X}$ is path-conn., we know there is a path $f: [0,1] \to \tilde{X}$ from x to y. It is obvious that $p \circ f$ is a path in X from $p(x)$ to $p(y)$.
Let $a \in A$ be as in the problem statement.
By path-connectedness of A, we can find paths α, β in A with $\alpha(0) = a = \beta(1)$ and $\alpha(1) = p(x)$ and $\beta(0) = p(y)$.
Then the product $\alpha \cdot (p \circ f) \cdot \beta$ is a loop in X at a. Since the map i_* from the problem statement is surjective, there exists a loop $\varphi: [0,1] \to A$ with $[\varphi] = [\alpha \cdot (p \circ f) \cdot \beta]$ (these equivalence classes are in the sense of of $\pi_1(X, a)$.).

· Now we will exploit the lifting properties of covering spaces.
Let $\tilde{\alpha}$ be a lift of α to a path in $\tilde{A}$ with $\tilde{\alpha}(1) = x$ (remarking that $x \in p^{-1}(p(x))$).
Let $w := \tilde{\alpha}(0)$. Let $\tilde{\beta}$ be a lift of β to a path in $\tilde{A}$ with $\tilde{\beta}(0) = y$.
Let $z := \tilde{\beta}(1)$. To summarize, $\tilde{\alpha}$ is a path from w to x and $\tilde{\beta}$ is a path from y to z.
We can also define liftings (in terms of $p: \tilde{X} \to X$) of α, β to paths in X beginning at w and y, respectively.
But since we can view the already defined $\tilde{\alpha}$ and $\tilde{\beta}$ as paths in $\tilde{X}$, uniqueness of path liftings gives us that these "new" lifts are exactly $\tilde{\alpha}$ and $\tilde{\beta}$. In other words, we obtain the same paths whether we lift in the sense of p or in the sense of the restriction of p to $p^{-1}(A)$.
Because φ is a loop in X, it lifts to a path $\tilde{\varphi}$ in $\tilde{X}$ with $p \circ \tilde{\varphi} = \varphi$ and $\tilde{\varphi}(0) = w$. In much the same way as with the lifts of α and β, we can see that $\tilde{\varphi}$ is a path in $\tilde{A}$.
Since path homotopies also lift to covering spaces, there is a path →

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homotopy in $\widetilde{X}$ from $\tilde{\alpha} \cdot f \cdot \tilde{\beta}$ to $\tilde{\varphi}$.

This establishes that $\tilde{\varphi}(1) = z$.

Finally, notice that $\bar{\tilde{\alpha}} \cdot \tilde{\varphi} \cdot \bar{\tilde{\beta}}$ is a path from x to y that is contained entirely in $\widetilde{A} = p^{-1}(A)$.

It follows that $p^{-1}(A)$ is path-connected. $\square$