...ary 2018

Let X be a topological space. For any subset A of X, is it always true that $X \setminus \bar{A} = \text{int}(X \setminus A)$? Prove your assertion. (Here $\bar{A}$ denotes the closure of A and $\text{int}(B)$ denotes the set of interior points of a set B.)

Pf: Observe that

$$X \setminus \bar{A} = X \setminus \Big(\bigcap_{\substack{K \supseteq A \\ K \text{ closed}}} K \Big) = \bigcup_{\substack{K \supseteq A \\ K \text{ closed}}} (X \setminus K) = \bigcup_{\substack{U \subseteq A^c \\ U \text{ open}}} U = \text{Int}(A^c) = \text{Int}(X \setminus A)$$

using the defn. of closure

using the DeMorgan laws

Since $A \subseteq K$, we have that $K^c \subseteq A^c$. Let $U = X \setminus K = K^c$

by defn. of interior

Therefore, we conclude that $X \setminus \bar{A} = \text{int}(X \setminus A)$. $\square$

OR

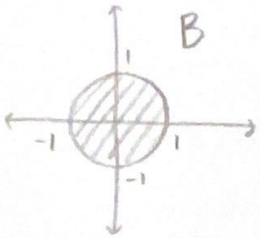
Pf:
- If $x \in X \setminus \bar{A}$, then $\exists$ an open nbhd U of x s.t. $U \cap A = \emptyset \Rightarrow U \subseteq A^c$. So $x \in U \subseteq A^c \Rightarrow x \in \text{Int}(A^c)$. Therefore, $X \setminus \bar{A} \subseteq \text{Int}(A^c) = \text{Int}(X \setminus A)$.
- If $x \in \text{Int}(A^c)$, then $\exists$ an open nbhd V of x s.t. $x \in V \subseteq A^c \Rightarrow V \cap A = \emptyset$. x has a nbhd disjoint from A, so $x \notin \bar{A} \Rightarrow x \in X \setminus \bar{A}$. Therefore, $\text{Int}(X \setminus A) \subseteq X \setminus \bar{A}$.

Thus, we conclude that $X \setminus \bar{A} = \text{Int}(X \setminus A)$. $\square$

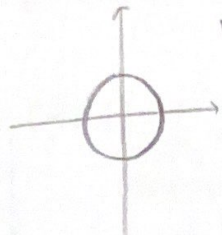
continued...

② Let $B = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 \leq 1\}$, $p = (1/2, 0)$, and $q = (-1/2, 0)$. Denote $M = B \setminus \{p,q\}$. Is M homotopic to the boundary of B? Prove your assertion.

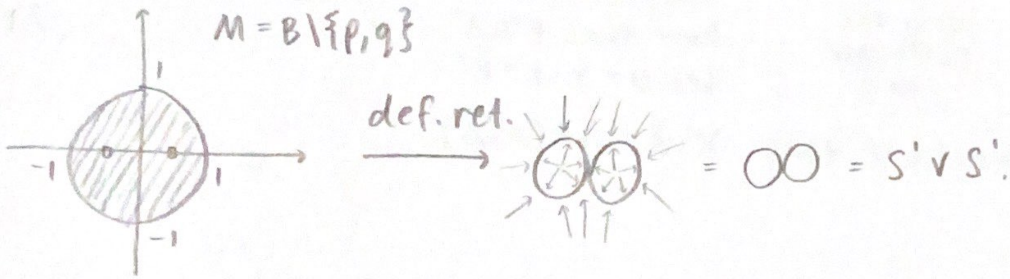
Pf:



The boundary of B is the boundary of the unit circle which is S^1.



$M = B \setminus \{p,q\}$



We have that $\pi_1(\partial B) = \pi_1(S^1) = \mathbb{Z}$ and that $\pi_1(M) = \pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$.

Therefore, we conclude that M cannot be homotopic to the boundary of B because they have different fundamental groups. $\square$

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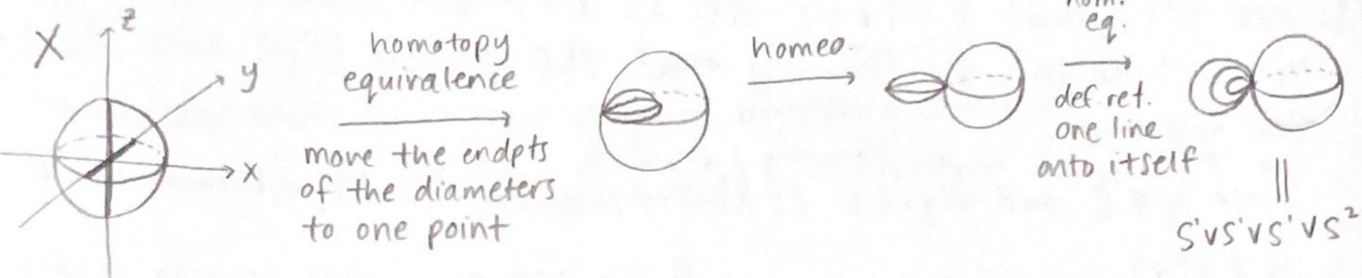
Let X be the union of the unit sphere $S^2 \equiv \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2=1\}$ with the two line segments $\{(0,y,0);\ |y| \le 1\} \cup \{(0,0,z);\ |z| \le 1\}$. Compute the fundamental group of X based at $(0,1,0)$.

Pf:



Since X is path-connected, the fundamental group is independent of the base point (up to isomorphism).

So $\pi_1(X) = \pi_1(S^1 \vee S^1 \vee S^1 \vee S^2)$.

Each space, S^1 and S^2, is locally Euclidean, so the wedge point has a nbhd in each space that def. ret. to the wedge point.

Therefore, we can use the following version of Van-Kampen:

$$\begin{aligned}\pi_1(S^1 \vee S^1 \vee S^1 \vee S^2) &= \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^2) \\ &= \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * 0 \\ &= \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.\end{aligned}$$

Thus, we conclude that $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. $\square$

continued...

(4) Let E be a subset of a topological space Y. Suppose that $f: Y \to E$ is a continuous map such that $f(x) = x$ for all $x \in E$. Show that if Y is Hausdorff, then E is a closed subset of Y.

Pf: We WTS that $E = \{x \in Y : f(x) = x\}$ is a closed subset of Y.

We will do this by showing that $Y \setminus E$ is open.

Let $y \in Y \setminus E$.

Since $y \notin E$ and $f(y) \in E$ (E is the codomain of f), we have that $y \neq f(y)$.

Since Y is Hausdorff and $y \neq f(y)$, we have that there exist open nbhds U of y and V of $f(y)$ s.t. $U \cap V = \emptyset$. ($U, V \subseteq Y$)

Consider $W = U \cap f^{-1}(V)$.

W is open because f is continuous, and $W \neq \emptyset$ because $y \in W$.

We WTS that $W \cap E = \emptyset$:

Let $x \in W = U \cap f^{-1}(V)$.

Then $x \in U$ and $f(x) \in f(f^{-1}(V)) = V$, but $U \cap V = \emptyset \Rightarrow x \neq f(x)$.

This means that $x \notin E$, since f fixes each point of E.

Since x was an arbitrary point of W, we have $W \cap E = \emptyset$.

So we have $y \in \overset{\text{open}}{W} \subseteq Y \setminus E$.

Therefore, $Y \setminus E$ is open $\Rightarrow E$ is closed.

$\square$

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Let $Mat_2(\mathbb{R})$ be the set of 2×2 real matrices with the topology obtained by regarding $Mat_2(\mathbb{R})$ as $\mathbb{R}^4$. Let $SO(2)=\{A\in Mat_2(\mathbb{R}); A^TA=I_2, \det A=1\}$ where A^T denotes the transpose of A, and I_2 is the 2×2 id. matrix.

(i) Show that $SO(2)$ is compact.

Pf: It is clear that $SO(2)$ only contains invertible matrices.
Since $A^{-1}=A^T$, we see that if $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SO(2)$, then $\begin{pmatrix} a & c \\ b & d\end{pmatrix}=\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$,

which means that $d=a$ and $c=-b$.
With the determinant condition in mind, $1=\det\begin{pmatrix} a & b \\ -b & a\end{pmatrix}=a^2+b^2$.
From here on, we will just write elements of $SO(2)$ as ordered quadruples and just write SO to denote this space.
We see that $SO=\{(a,b,-b,a)\in\mathbb{R}^4 : a^2+b^2=1\}$.
There is an obvious continuous map from SO to S^1, namely f defined by $f(a,b,-b,a)=(a,b)$.
This is cts b/c each component function is projection.
To see that f is bijective, notice that $g: S^1\to SO$ defined by $g(a,b)=(a,b,-b,a)$ is a suitable inverse.
It is continuous b/c each component is either projection or projection composed w/ multiplication by -1.
Hence, f is a homeomorphism between SO and S^1.
Thus, SO is compact since S^1 is a closed and bdd subset of $\mathbb{R}^2$. $\square$

(ii) Is $SO(2)$ connected? Prove your assertion.

Pf: Yes, $SO(2)$ is connected because S^1 is path-connected, which implies that $SO(2)$ is path-connected, which implies that $SO(2)$ is connected. ($SO(2)$ and S^1 are homeomorphic).

$\square$

Continued.

⑥ Find a simply-connected covering space for the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$. Justify your reasoning. (The space $\mathbb{RP}^2$ is the quotient space of the unit sphere S^2 obtained by identifying the antipodal points.)

Pf. Observe that the space $\mathbb{RP}^2 \# \mathbb{RP}^2$ is a Klein bottle $= K$.

Notice that the Klien bottle has a two-sheeted covering by the torus.

Let $p: S^1 \times S^1 \to K$ denote this covering. Since the product of covering maps is a covering map, there is a covering map $q: \mathbb{R}^2 \to S^1 \times S^1$.

Since p is a finite sheeted cover, we know that the composition $p \circ q$ is a covering map.

Hence $\mathbb{R}^2$ is a covering space of K.

Since $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong K$, we have that $\mathbb{R}^2$ is the universal cover of $\mathbb{RP}^2 \# \mathbb{RP}^2$.

$\square$