...ary 2019

1) Let X be the subset $(\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})\subseteq\mathbb{R}^2$. Define an equivalence relation on X by declaring $(x,0)\sim(x,1)$ if $x\neq 0$. Show that the quotient space $X\backslash\sim$ is not Hausdorff.

Pf: Let $q: X\to X\backslash\sim$ be the quotient map.

Consider the points $(0,0)$ and $(0,1)$.

We WTS that every nbhd of $(0,0)$ intersects every nbhd of $(0,1)$.

Let $U\subseteq X\backslash\sim$ be an open nbhd of $(0,0)$. Then $q^{-1}(U)$ is open in X.

We know that $(0,0)\in q^{-1}(U)$ and since $q^{-1}(U)$ is open

$(0,0)\in\{(x,0):|x|<a\}\subseteq q^{-1}(U)$

and $\{(x,1):0<|x|<a\}\subseteq q^{-1}(U)$.

Let $V\subseteq X\backslash\sim$ be an open nbhd of $(0,1)$. Then $q^{-1}(V)$ is open in X.

We know that $(0,1)\in q^{-1}(V)$ and since $q^{-1}(V)$ is open

$(0,1)\in\{(x,1):|x|<b\}\subseteq q^{-1}(V)$

and $\{(x,0):0<|x|<b\}\subseteq q^{-1}(V)$.

Let $0<c<\min\{a,b\}$.

Then $(c,1)\in q^{-1}(U)\cap q^{-1}(V)=q^{-1}(U\cap V)$.

If $U\cap V=\emptyset$, then $q^{-1}(U\cap V)=\emptyset$.

Since $(c,1)\in q^{-1}(U\cap V)\neq\emptyset$, we have that $U\cap V\neq\emptyset$.

Therefore, we have shown that every nbhd of $(0,0)$ intersects every nbhd of $(0,1)$.

Thus, $X\backslash\sim$ is not Hausdorff. $\square$

continued...

② Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be locally finite if each point of X has a nbhd that intersects at most finitely many of the sets in $\mathcal{A}$. Show that if $\mathcal{A}$ is a locally finite collection of subset of X, then $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}$.

Pf: • Let $x \in \bigcup_{A\in\mathcal{A}} \overline{A}$. Then $x \in \overline{A}$ for some $A \in \mathcal{A}$.

Since $x \in \overline{A}$, every nbhd of x intersects A.

Observe that $A \subseteq \bigcup_{A\in\mathcal{A}} A$.

So every nbhd of x intersects $\bigcup_{A\in\mathcal{A}} A$.

Therefore, $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$.

• Let $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$.

Then every nbhd of x intersects $\bigcup_{A\in\mathcal{A}} A$. Since $\mathcal{A}$ is locally finite, we know that $\exists$ a nbhd U of x s.t. U intersects finitely many $A_i \in \mathcal{A}$ (sets in $\mathcal{A}$): U intersects $\bigcup_{i=1}^{n} A_i$, where $A_i \in \mathcal{A}$.

Assume $x \notin \bigcup_{i=1}^{n} \overline{A_i}$. Then there are nbhds V_i of x s.t. $V_i \cap A_i = \emptyset$ for $i=1,\dots,n$. Let $V = \bigcap_{i=1}^{n} V_i$. So we have $x \in V$ and V is disjoint from $A_1,\dots,A_n$.

Consider $U \cap V$. Observe that $x \in U \cap V$ and $U \cap V$ is open since it is the intersection of finitely many open sets.

Therefore, $U \cap V$ is a nbhd of x that does not intersect any $A \in \mathcal{A}$. ↯

This is a contradiction because every nbhd of x is supposed to intersect $\bigcup_{A\in\mathcal{A}} A$.

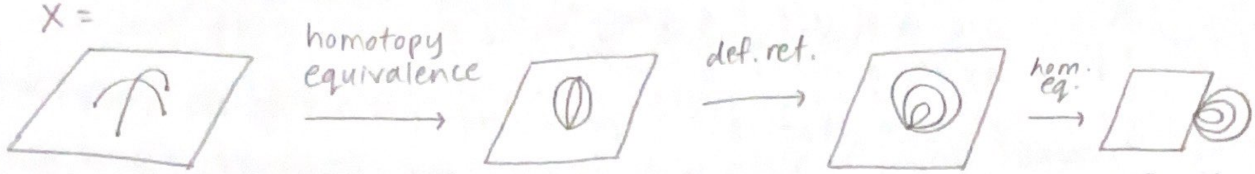
Thus, we conclude that $x \in \bigcup_{A\in\mathcal{A}} \overline{A}$.

Therefore, $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}$. $\square$

...ued...

Let $X = (\mathbb{R}^2 \times \{0\}) \cup \{(0,y,z) : y^2+z^2=1, z \geq 0\} \cup \{(x,0,z) : x^2+z^2=1, z\geq 0\}$.
Compute the fundamental group of X based at $(0,0,0)$.

Pf: $X =$



$= \mathbb{R}^2 \vee S^1 \vee S^1 \vee S^1$

Therefore, $\pi_1(X) = \pi_1(\mathbb{R}^2 \vee S^1 \vee S^1 \vee S^1)$ where $\mathbb{R}^2$ and S^1 are path-conn.

Thus, $\pi_1(X) = \pi_1(\mathbb{R}^2) * \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^1)$

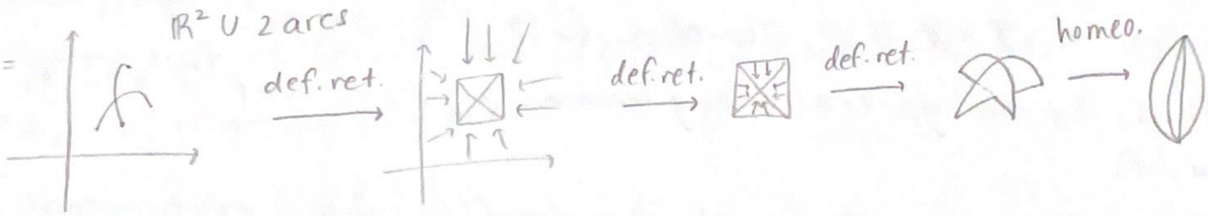
$= 0 * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$= \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

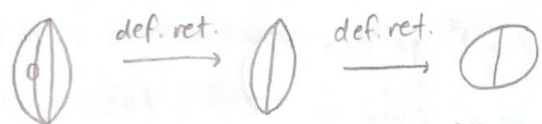
$\square$

OR

Pf: $X =$



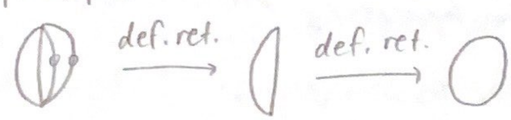
Let $U =$



$= \Theta$-space

U is open, path-connected. $\pi_1(U) = \pi_1(\Theta\text{-space}) = \mathbb{Z} * \mathbb{Z}$

Let $V =$

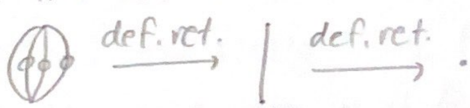


$= S^1$

V is open, path-connected. $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$

Observe that $X = U \cup V$.

$U \cap V =$



$U \cap V$ is path-conn., nonempty. $\pi_1(U \cap V) = \emptyset$.

Since $U \cap V$ is simply-conn., we can use the following version of Van Kampen,

$\pi_1(X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

$\square$

continued...

④ Let $\mathbb{P}^2$ denote the projective plane. Prove that any continuous map $f: \mathbb{P}^2 \to$
is null-homotopic, i.e., homotopic to a constant map.

Pf:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{P}^2 \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & \mathbb{T}^2 \end{tikzcd}$$

we will use the general lifting lemma to show that there exists a lift $\tilde{f}: \mathbb{P}^2 \to \mathbb{R}^2$.

observe that $\mathbb{P}^2$ is path-conn. (it is the cts image of S^2 which is path-conn, and the cts image of path-conn. is path-conn.) and $\mathbb{P}^2$ is locally path-conn. (because $q: S^2 \to \mathbb{P}^2$ is a local homeo.).

Let $p: \mathbb{R}^2 \to \mathbb{T}^2$ be a covering map, where p is the product of two exp. maps.

To use the general lifting lemma, it remains to show that

$f_*(\pi_1(\mathbb{P}^2)) \subseteq p_*(\pi_1(\mathbb{R}^2))$.

Observe that $\pi_1(\mathbb{R}^2) = 0$, and $\pi_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z}$, which is finite, so $f_*(\pi_1(\mathbb{P}^2))$ is finite. we WTS $f_*(\pi_1(\mathbb{P}^2)) = 0$.

We know that $f_*(\pi_1(\mathbb{P}^2)) \subseteq \pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$, and the only finite subgp of $\mathbb{Z} \times \mathbb{Z}$ is 0. Therefore, $f_*(\pi_1(\mathbb{P}^2)) = 0 \subseteq p_*(\pi_1(\mathbb{T}^2))$. ✓

Thus, by the general lifting lemma, we have that $\tilde{f}: \mathbb{P}^2 \to \mathbb{R}^2$ is a lift.

Recall that any cts fn. into a contractible space is null-homotopic.

Since $\tilde{f}$ is cts and $\mathbb{R}^2$ is contractible, we have that $\tilde{f}$ is null-homotopic.

Since $\tilde{f}$ is null-homotopic, so is f (let H be the homotopy btwn $\tilde{f}$ and a constant map, then $p \circ H$ is a homotopy btwn f and a constant map.). Therefore, f is null-homotopic.

□

ued.

Let $U = \mathbb{R}^2 \setminus S = \{x \in \mathbb{R}^2 : x \notin S\}$, where $S \subset \mathbb{R}^2$ is a countable set. Is U path-connected? Justify your answer.

Pf: We will prove that U must be path-connected.

Let $x, y \in U$. We want to find a path p from x to y.

First observe that there cannot be a circle of points removed around a point. There is a direction for each $\theta \in [0, 2\pi)$ starting at a point, so there are uncountably many directions around that point, but only countably many poinst removed (i.e, in S).

Therefore, we can form a path out of each point.

Let $L_m(x)$ = the line through x with slope $m \subseteq \mathbb{R}^2$.

Observe that $L_m(x) \cap L_{m'}(x) = \{x\}$ $(m \neq m')$

Each point in S lies on $L_m(x)$ for at most one $m \in \mathbb{R}$.

We WTS that $\exists\, m$ s.t. $L_m(x)$ is disjoint from S.

Suppose no such m exists. Then $\forall m \in \mathbb{R}$, $\exists\, x_m \in S \cap L_m(x)$.

$x_m \neq x_{m'}$ by what we said earlier.

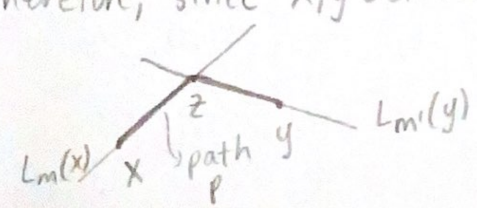
Therefore, $\underbrace{\{x_m : m \in \mathbb{R}\}}_{\text{uncountable}} \subseteq \underbrace{S}_{\text{countable}}$. $\lightning$

This is a contradiction. So $\exists\, m$ s.t. $L_m(x)$ is disjoint from S (in fact, countably many such lines).

By the same argument, there are two lines through y s.t. they do not intersect S. In particular, there is a line $L_{m'}(y)$ s.t. $m \neq m'$ and $L_{m'}(y) \cap S = \emptyset$.

If we let z be the intersection point of $L_m(x)$ and $L_{m'}(y)$, then our path is: the straight line from x to z, and the straight line from z to y is a path in U from x to y.

Therefore, since $x, y \in U$ were arbitrary, we have that U is path-conn. $\square$



Continued...

⑥ Let X be a topological space and $q: \mathbb{R}^2 \to X$ be a covering map. Let $B = \{(x,y) : x^2+y^2 \leq 1\} \subset \mathbb{R}^2$ and let K be a compact subset of X. Suppose $q: \mathbb{R}^2 \setminus B \to X \setminus K$ is a homeomorphism. Show that X is homeomorphic to $\mathbb{R}^2$.

Pf: Non-deck transformation soln:

Goal: · Show q is injective (injective covering map = homeo.)
or · Show X is simply connected (any covering from a path-conn. space to a simply conn. space is homeo.)

We will show q is injective.

K is closed: we know B is closed, so $\mathbb{R}^2 \setminus B$ is open. q is open, so $q(\mathbb{R}^2 \setminus B) = X \setminus K$ is open $\Rightarrow K$ is closed.

$q^{-1}(K)$ is closed (b/c q is continuous).

We have that $q^{-1}(K) \subseteq B$, where B is compact.

$q^{-1}(K)$ is a closed subset of a compact set $\Rightarrow q^{-1}(K)$ is compact.

We have that $q: \underset{\text{saturated}}{q^{-1}(K)} \to K$ is a covering map (b/c q restricted to $q^{-1}(K)$)

$q_K: \underset{\text{compact}}{q^{-1}(K)} \to \underset{\text{compact}}{K}$ covering map with compact domain has finite fibers.
compact $\Rightarrow$ compact

$q: \mathbb{R}^2 \to X$ connected covering, so all fibers have the same size (cardinality) which is finite, so $\#(q^{-1}(x)) = n \ \forall x \in X$.

$q(B) \supseteq K$

If $\underset{\text{cpt}}{q(B)} = \underset{\text{cpt}}{X}$ $\quad$ q is a covering map with finite fibers and cpt codomain
cpt $\Rightarrow$ cpt $\quad$ $\Rightarrow$ domain is cpt $\to \mathbb{R}^2$ is not cpt. ↯

So $\exists x \in X$ s.t. $x \notin q(B)$. $\quad \underbrace{q^{-1}(x)}_{\text{nonempty}} \subseteq \mathbb{R}^2 \setminus B$.

$q: \mathbb{R}^2 \setminus B \to X \setminus K$ is injective

one point $= q^{-1}(x) \cap \mathbb{R}^2 \setminus B = q^{-1}(x)$

$\#(q^{-1}(x)) = 1$, connected covering, q is injective.

$\square$

ued.

Deck transformation soln:

$q: \mathbb{R}^2 \to X$ covering map.

$Aut_q(\mathbb{R}^2)$ is the group of homeomorphisms $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$q(x) = q \circ \varphi(x) \quad \forall\, x \in \mathbb{R}^2$.

Two facts:

① Since $\mathbb{R}^2$ simply connected, $\pi_1(X) \cong Aut_q(\mathbb{R}^2)$

② If φ, ψ are in $Aut_q(\mathbb{R}^2)$ and $\varphi(x) = \psi(x)$ for $x \in \mathbb{R}^2$, then $\varphi = \psi$ identically.

Our goal: Show $Aut_q(\mathbb{R}^2)$ is trivial, by fact ① this means X is simply connected.

Identity element of $Aut_q(\mathbb{R}^2)$ is $id_{\mathbb{R}^2}$.

Let $\varphi \in Aut_q(\mathbb{R}^2)$ be arbitrary.

If $\varphi(x) = x$, for any $x \in \mathbb{R}^2$, by ② $\varphi = id_{\mathbb{R}^2}$.

Let $y \in \mathbb{R}^2 \setminus B$.

If $\varphi(y) = y$, then φ is the identity.

$\varphi(y) \in B$ (because $q: \mathbb{R}^2 \setminus B \to X \setminus K$ is injective)

$q \circ \varphi(y) = q(y)$

This holds $\forall y \in \mathbb{R}^2 \setminus B$, so $\varphi(\mathbb{R}^2 \setminus B) \subseteq B$ b/c φ is bijective

$$\varphi(B) \supseteq \mathbb{R}^2 \setminus B$$

$$\underbrace{\mathbb{R}^2 \setminus B}_{\text{not bounded}} \subseteq \underbrace{\varphi(B)}_{\substack{\text{compact} \\ \text{closed and bounded}}} \quad ↯$$

$\exists\, y \in \mathbb{R}^2 \setminus B$ w/ $\varphi(y) = y \Rightarrow \varphi = id_{\mathbb{R}^2}$ by fact ②.

□