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Let X be a topological space. Prove or disprove the following assertions.

1. Let $A_1, \dots, A_K$ be subsets of X. Then $\overline{\bigcup_{i=1}^{K} A_i} = \bigcup_{i=1}^{k} \overline{A_i}$.

Pf. We will prove that $\overline{\bigcup_{i=1}^{K} A_i} = \bigcup_{i=1}^{K} \overline{A_i}$ for $A_1, \dots, A_K \subseteq X$ by induction on K.

If $k=1$, then $\overline{A_1} = \overline{A_1}$.

- So let $k=2$ be our base case: we WTS $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.
  - First we will show that $\overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}$.

    Observe that $A_1 \subseteq A_1 \cup A_2 \subseteq \overline{A_1 \cup A_2}$, so $\overline{A_1 \cup A_2}$ is a closed set containing A_1. We know that $\overline{A_1}$ is the smallest closed set containing A_1. Therefore, $\overline{A_1} \subseteq \overline{A_1 \cup A_2}$.

    Likewise, observe that $A_2 \subseteq A_1 \cup A_2 \subseteq \overline{A_1 \cup A_2}$, so $\overline{A_1 \cup A_2}$ is a closed set containing A_2. We know that $\overline{A_2}$ is the smallest closed set containing A_2. Therefore, $\overline{A_2} \subseteq \overline{A_1 \cup A_2}$.

    Thus, we have that $\overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}$.
  - Now we will show that $\overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}$.

    We will do this by showing that if $x \notin \overline{A_1} \cup \overline{A_2}$, then $x \notin \overline{A_1 \cup A_2}$.

    Suppose $x \notin \overline{A_1} \cup \overline{A_2}$.

    If $x \notin \overline{A_1}$, then $\exists$ an open nbhd U_1 of x s.t. $U_1 \cap A_1 = \emptyset$.

    If $x \notin \overline{A_2}$, then $\exists$ an open nbhd U_2 of x s.t. $U_2 \cap A_2 = \emptyset$.

    Observe that $U_1 \cap U_2$ is open (the finite intersection of open sets is open) and that $U_1 \cap U_2$ is a nbhd of x ($x \in U_1 \cap U_2$).

    Observe that $(U_1 \cap U_2) \cap (A_1 \cup A_2) = \emptyset$. Therefore, $x \notin \overline{A_1 \cup A_2}$ b/c $U_1 \cap U_2$ is an open nbhd of x that is disjoint from $A_1 \cup A_2$.

    Thus, we have that $\overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}$.

  We conclude that $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.
- Assume that $\overline{A_1 \cup \dots \cup A_{K-1}} = \overline{A_1} \cup \dots \cup \overline{A_{K-1}}$ (Inductive hypothesis).

  We WTS that $\overline{A_1 \cup \dots \cup A_{K-1} \cup A_K} = \overline{A_1} \cup \dots \cup \overline{A_{K-1}} \cup \overline{A_K}$.

  Let $B = A_1 \cup \dots \cup A_{K-1}$. Then $\overline{B \cup A_K} = \overline{B} \cup \overline{A_K}$ by our base case.

  The LHS is $\overline{B \cup A_K} = \overline{A_1 \cup \dots \cup A_{K-1} \cup A_K}$.

continued...

The RHS is $\overline{B} \cup \overline{A_K} = \overline{A_1 \cup \ldots \cup A_{K-1}} \cup \overline{A_K}$) by the induction hypothesis.

$= \overline{A_1} \cup \ldots \cup \overline{A_{K-1}} \cup \overline{A_K}$

Therefore, $\overline{A_1 \cup \ldots \cup A_{K-1} \cup A_K} = \overline{A_1} \cup \ldots \cup \overline{A_{K-1}} \cup \overline{A_K}$

i.e., $\overline{\bigcup_{i=1}^{K} A_i} = \bigcup_{i=1}^{K} \overline{A_i}$ for $A_1, \ldots, A_K \subseteq X$. $\square$

2. Let $\{B_i\}_{i=1}^{\infty}$ be subsets of X. Then $\overline{\bigcup_{i=1}^{\infty} B_i} = \bigcup_{i=1}^{\infty} \overline{B_i}$.

Pf: This statement is false.

Let $X = \mathbb{R}$ and consider the subsets $\{B_i\}_{i=1}^{\infty} = \{(\frac{1}{i}, 1)\}_{i=1}^{\infty}$.

Then $\overline{B_i} = [\frac{1}{i}, 1]$ for $i=1$ to ∞.

So $\bigcup_{i=1}^{\infty} \overline{B_i} = (0, 1]$

On the other hand, we have $\overline{\bigcup_{i=1}^{\infty} B_i} = [0, 1]$.

It is clear that $(0,1] \neq [0,1]$.

Therefore, we conclude that $\overline{\bigcup_{i=1}^{\infty} B_i} \neq \bigcup_{i=1}^{\infty} \overline{B_i}$ for $\{B_i\}_{i=1}^{\infty} \subseteq X$.

$\square$

To simplify one direction of 1.a):

the union of two closed sets is closed.

So $\overline{A_1} \cup \overline{A_2}$ is a closed set containing $A_1 \cup A_2$

Hence $\overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}$.

...ued.

Let (X,d) be a complete metric space and $\{E_i\}_{i=1}^{\infty}$ be a sequence of nonempty closed subsets so that $E_{i+1} \subseteq E_i$ for all i. Suppose the diameter $\operatorname{diam}(E_i) \to 0$ as $i \to \infty$. Show that $\bigcap_{i=1}^{\infty} E_i$ is nonempty and consists of precisely one point. (Recall that the diameter of a metric space E is defined by $\operatorname{diam}(E) = \sup\{d(x,y) : x,y \in E\}$.)

Pf: We WTS that $\bigcap_{i=1}^{\infty} E_i$ contains exactly one point.

- First we will show that $\bigcap_{i=1}^{\infty} E_i$ has at most one point (uniqueness).
  Assume $\exists\, x,y \in E$ distinct. Then $d(x,y) > 0$.
  We have that $\operatorname{diam}(E_i) \geq d(x,y)$ $\forall i$ because $x,y \in E_i$ $\forall i$
  So $0 = \lim_{i\to\infty} \operatorname{diam}(E_i) \geq d(x,y) > 0$ $\lightning$.
  Therefore, $\bigcap_{i=1}^{\infty} E_i$ must have at most one point.

- Now we will show that $\bigcap_{i=1}^{\infty} E_i$ has at least one point (existence).
  Let $\{x_i\}$ be any sequence with $x_i \in E_i$.
  Fix $\varepsilon > 0$. Let N be such that $\operatorname{diam}(E_i) < \varepsilon$ $\forall i > N$.
  Let $k > j$, then the nested condition tells us that $x_j, x_k \in E_j$.
  We have that $d(x_j, x_k) \leq \operatorname{diam}(E_j) < \varepsilon$ as long as $j > N$.
  For $j,k > N$, $\min(j,k) > N$ implies $d(x_j,x_k) < \varepsilon$.
  Therefore, $\{x_i\}$ is a Cauchy sequence.
  Since $\{x_i\}$ is a Cauchy sequence, we have that $x_i \to x$ for some $x \in X$ because (X,d) is complete.
  We WTS $x \in E$.
  Observe that $E_1 : \{x_i\}_{i=1}^{\infty} \subseteq E_1$, so closedness of E_1 tells us $x \in E_1$.
  $\vdots$
  $E_n : \{x_i\}_{i=n}^{\infty} \subseteq E_n$, $\lim_{i\to\infty} x_i = x$, so $x \in E_n$ by closedness.

  Therefore, $x \in E_n$ $\forall n$ implies that $x \in E$.

- Thus, since $\bigcap_{i=1}^{\infty} E_i$ contains at most and at least one point, it must consist of precisely one point.

$\square$

continued...

(3) Let Z be the topology on $\mathbb{R}^2$ such that every nonempty open set of Z is of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many points}\}$. Show that any continuous function $f: (\mathbb{R}^2, Z) \to \mathbb{R}$ is constant, where $\mathbb{R}$ is endowed with the standard topology.

Pf: Let Y be a Hausdorff space.

We will show that $f: (\mathbb{R}^2, Z) \to Y$ continuous, is constant.

(i.e., $f((x,y)) = c$ for some constant $c \in Y$).

Assume that f is continuous and nonconstant.

Since f is nonconstant, we know that there exist distinct $a, b \in f(\mathbb{R}^2, Z) \subseteq Y$.

Since Y is Hausdorff, we know that there exist open nbhds U of a and V of b s.t. $U \cap V = \emptyset$.

Since f is continuous and $U, V \subseteq Y$ are open, we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open in $(\mathbb{R}^2, Z)$.

Observe that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$ since $U \cap V = \emptyset$.

Since $f^{-1}(U), f^{-1}(V)$ are open and nonempty in $(\mathbb{R}^2, Z)$, they must be of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many points}\}$.

Suppose $f^{-1}(U) = \mathbb{R}^2 \setminus \{a_1, \dots, a_n\}$ and $f^{-1}(V) = \mathbb{R}^2 \setminus \{b_1, \dots, b_m\}$.

Then $f^{-1}(U) \cap f^{-1}(V) = \mathbb{R}^2 \setminus \{a_1, \dots, a_n, b_1, \dots, b_m\}$ ← finitely many points

$\neq \emptyset$ b/c $\mathbb{R}^2$ is infinite. ↯

This contradicts $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Therefore, f must be constant.

Since $\mathbb{R}$ is Hausdorff, we have shown that any continuous function $f: (\mathbb{R}^2, Z) \to \mathbb{R}$ is constant.

$\square$

ued.

Let D^2 be a closed disk in $\mathbb{R}^2$ and S^1 be the boundary unit circle. Prove or disprove the following statements.

1. Let $f: S^1 \to D^2$ be a continuous map. Then f extends to a continuous map $F: D^2 \to D^2$ with $F|_{S^1} = f$.

Pf: This statement is true.

Observe that D^2 is convex $\Rightarrow$ contractible.

A continuous function into a contractible space is null-homotopic.

Since $f: S^1 \to D^2$ is cts and D^2 is contractible $\Rightarrow$ f is null-homotopic.

Let $H: [0,1] \times S^1 \to D^2$ be a homotopy between f and a constant map.

We want a map $F: D^2 \to D^2$.

Observe that $D^2 = \bigcup_{r \in [0,1]} S_r$, where S_r is a circle of radius r.

D^2 is homeomorphic to $\underbrace{([0,1] \times S^1)/(S^1 \times \{0\})}_{\text{this is a cone which is homeo. to a disk}}$ (because S_0 is just a point)

H is a homotopy from f to a constant map: $H(0,s) = s_0$ and $H(1,s) = f(s)$.

So $H(0,s) = s_0$ is the same for every s.

H respects the quotient identification because $\{0\} \times S^1$ gets collapsed to a point and H is constant on $\{0\} \times S^1$. So H descends to a map out of the quotient, call this map F.

Then $F(1,s) = H(1,s) = f(s)$, so F extends to f. $\square$

2. There is a map $g: D^2 \to S^1$ such that $g|_{S^1}$ is the identity map on S^1.

Pf: This statement is false.

If $g: D^2 \to S^1$ were a retract, then the induced homomorphism $g_*: \pi_1(D^2) \to \pi_1(S^1)$ is surjective.

But $\pi_1(S^1) = \mathbb{Z}$ and the fundamental group of D^2 is trivial.

Therefore, g_* is not surjective $\Rightarrow$ g is not a retract.

Thus, there is no such g. $\square$

continued...

⑥ Let X, Y, Z be convex open subsets in $\mathbb{R}^n$, $n \geq 1$. Suppose $X \cap Y \cap Z \neq \emptyset$. Show that their union $X \cup Y \cup Z$ is simply-connected.

Pf: Let $a \in X \cap Y \cup Z$. (since $X \cap Y \cap Z \neq \emptyset$).

Observe that since X, Y, Z are convex:

$\forall x \in X$, the line joining x to a lies in X, so it lies in $X \cup Y \cup Z$.

$\forall y \in Y$, the line joining y to a lies in Y, so it lies in $X \cup Y \cup Z$.

$\forall z \in Z$, the line joining z to a lies in Z, so it lies in $X \cup Y \cup Z$.

$X \cup Y \cup Z$ is star-convex with center a.

Star-convex spaces are simply-connected:

Let f be any loop based at a. $\pi_1(X \cup Y \cup Z, a)$ is trivial.

Define $H(s,t)$ by $H(s,t) = tf(s) + (1-t)a$.

Then $H(s,0) = a$

$H(s,1) = f(s)$

$H(0,t) = ta + (1-t)a = a$

$H(1,t) = ta + (1-t)a = a$

Observe that $H(s,t) \in X \cup Y \cup Z$ because $H(s,t)$ lies on the straight line path from $f(s)$ to a.

Therefore, we conclude that $X \cup Y \cup Z$ is simply-connected.

$\square$