ary 2021

Suppose X and Y are topological spaces, and $f: X \to Y$ is a map satisfying $\operatorname{Int} f^{-1}(B) \subseteq f^{-1}(\operatorname{Int} B)$ for all $B \subseteq Y$. Is f an open map? Prove your assertion.

Pf: We want to show that f is open: if $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.

Let $U \subseteq X$ be open, and let $B = f(U) \subseteq Y$.

We WTS that B is open.

Let $y \in B$. We WTS that $y \in \operatorname{Int}(B)$, because then $B = \operatorname{Int}(B)$, ($\operatorname{Int}(B) \subseteq B$)

So we have that B is open.

Observe that $f^{-1}(B) = f^{-1}(f(U)) \supseteq U$.

Since U is open, we have that $U \subseteq \operatorname{Int} f^{-1}(f(U)) = \operatorname{Int}(f^{-1}(B))$.

Therefore, we have that $U \subseteq \operatorname{Int}(f^{-1}(B)) \subseteq f^{-1}(\operatorname{Int}(B))$.

Since $y \in B = f(U)$, we know $\exists x \in U$ s.t. $f(x) = y$.

So $\exists x \in U \subseteq f^{-1}(\operatorname{Int}(B))$, which means that $f(x) \in \operatorname{Int}(B)$.

So $f(x) = y \in \operatorname{Int}(B)$.

Therefore, $B = \operatorname{Int}(B) \Rightarrow B = f(U)$ is open.

Thus, f is an open map. $\square$

continued...

② Let $L_n \subset \mathbb{R}^2$ denote the closed line segment joining the point $(0,0)$ to the point $(\frac{1}{n}, 1)$. Consider $X = \bigcup_{n=1}^{\infty} L_n \cup \{(0,1)\}$ with its subspace topology induced from $\mathbb{R}^2$.

(1) Is X connected? Prove your assertion.

Pf Yes.

Let $U, V \subseteq \mathbb{R}^2$ be open sets s.t. $X \cap U$ and $X \cap V$ are open in X.

Then WLOG, $(0,1) \in U$.

Since U is open, there is an open ball $B((0,1), r)$ contained in U.

Thus, there is an $n \in \mathbb{N}$ s.t. $\frac{1}{n} < r$ and thus $(\frac{1}{n}, 1) \in B((0,1), r)$.

This is a point on the line connecting $(0,0)$ to $(\frac{1}{n}, 1)$.

Thus, since a line segment is homeomorphic to $[0,1]$, it is connected, so U must contain all of it, including $(0,0)$.

Since $(0,0)$ is in each line segment of X, U must contain each L_n.

Thus, $X \subseteq U \Rightarrow X = U$, so X is connected. □

(2) Is X path-connected? Prove your assertion.

Pf: No.

Assume that X is path-connected.

Then there is a path γ connecting $(0,0)$ and $(0,1)$ so that $\gamma(0) = (0,0)$ and $\gamma(1) = (0,1)$.

Let $t_1 = \sup\{t \in [0,1] : \gamma(t) = (0,0)\}$ and $t_2 = \inf\{t \in [t_1, 1] : \gamma(t) = (0,1)\}$.

Then $\gamma\left(\frac{t_1+t_2}{2}\right) \neq (0,0), (0,1)$ since $t_1 < \frac{t_1+t_2}{2}$ and $t_2 > \frac{t_1+t_2}{2}$.

So $\gamma\left(\frac{t_1+t_2}{2}\right)$ is on some line connecting $(0,0)$ to $(\frac{1}{n}, 1)$ for some n.

But since $(0,1)$ is not on this line, γ must pass through $(0,0)$ at some point on the interval $\left[\frac{t_1+t_2}{2}, t_2\right]$ since $(0,0)$ is the only point connecting this line to the rest of X. ↯

This is a contradiction.

Thus, X is not path-connected. □

...ued...

Let X be a locally compact Hausdorff space. Let ∞ be some object not in X and consider $X^* = X \sqcup \{\infty\}$ with the following topology:

$\mathcal{T} = \{\text{open subsets of } X\} \cup \{U \subseteq X^* : X^* \setminus U \text{ is a compact subset of } X\}$.

(1) Show that X^* is a compact Hausdorff space.

Pf: • First we will show that X^* is compact.

Let $\{U_\alpha\}$ be an open cover of X^*. We WTS $\exists$ a finite subcover.

We know that at least one element of $\{U_\alpha\}$, say U_0, is in $\mathcal{T}_2 = \{U \subseteq X^* : X^* \setminus U \text{ is a cpt subset of } X\}$ because the only nbhds of ∞ are the elements of $\mathcal{T}_2$ since $\infty \notin X$ and $\mathcal{T}_1 = \{\text{open subsets of } X\}$.

If $U_0 \in \mathcal{T}_2$, then $X^* \setminus U_0$ is compact, so there exist $U_1, \dots, U_n$ that cover $X^* \setminus U_0$.

If we add U_0, then we get that $\{U_0, U_1, \dots, U_n\}$ is still finite and covers X^*.

Therefore, X^* is compact.

• Now we will show that X^* is Hausdorff.

Let $x, y \in X^*$ be distinct points.

We need U, V open disjoint nbhds of x, y, respectively $\left(\begin{smallmatrix} x \in U \subseteq \mathcal{T} \\ y \in V \subseteq \mathcal{T} \end{smallmatrix}, U \cap V = \emptyset\right)$

If $x \neq \infty$, $y \neq \infty$, then $x, y \in X$. Since X is Hausdorff, we know there exist open nbhds U of x and V of y s.t. $U \cap V = \emptyset$. ($U, V \in \mathcal{T}_1 \subseteq \mathcal{T}$).

If $x = \infty$ and $y \neq \infty$, then we have an open nbhd of x in $\mathcal{T}_2$ and an open nbhd of y in $\mathcal{T}_1$.

We want $x \in U \subseteq \mathcal{T}_2$, so $X^* \setminus U$ is compact, and let $y \in V \subseteq \mathcal{T}_1$.

We need an open nbhd V of y and $V \subseteq K$, where K is compact.

Since X is locally compact, there exists V a nbhd of y and K compact s.t. $y \in V \subseteq K$.

Let $U = X^* \setminus K$ be an open nbhd of ∞. It is disjoint from $V \subseteq K$ ($U \cap V = X^* \setminus K \cap V = \emptyset$, $V \subseteq K$).

Therefore, we conclude that X^* is Hausdorff.

Thus, X^* is a compact Hausdorff space.

$\square$

continued...

(2) Show that X is dense in X^* if and only if X is noncompact.

Pf: If X is noncompact, then $X^* \setminus X = \{\infty\}$ is not open.

So X is not closed $\Rightarrow X \subsetneq \overline{X}$, and $\overline{X} = X^*$.

Therefore, X is dense in X^*.

If X is compact, then $X^* \setminus X = \{\infty\}$ is open.

So X is closed $\Rightarrow \overline{X} = X \neq X^*$.

Therefore, X is not dense in X^*. $\square$

OR: If X is dense in X^*, then X is not closed.

$X^* \setminus X = \{\infty\}$ is not open.

X is not compact.

OR: In a compact Hausdorff space, subsets are closed iff compact.

So if X is dense in X^*, X is not closed, so X is not compact.

...ued...

Let $p: E \to X$ be a covering map with $p(e_0) = x_0$. The lifting correspondence is denoted by $\varphi: \pi_1(X, x_0) \to p^{-1}(x_0)$. Show that if E is simply connected, then φ is bijective.

Pf: Recall that for $\alpha \in \pi_1(X, x_0)$, $\Phi(\alpha) = \tilde{\alpha}_{e_0}(1)$.

- First we will show that φ is surjective:

Let $e_1 \in p^{-1}(x_0)$.

We want a loop $\alpha \in \pi_1(X, x_0)$ s.t. $\varphi(\alpha) = e_1 = \tilde{\alpha}_{e_0}(1)$.

Let β be any path in E from e_0 to e_1 (we can do this b/c E simply connected $\Rightarrow$ E is path-connected).

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[ur, "\beta"] \arrow[r, "p\circ\beta"'] & X \end{tikzcd}$$

$p \circ \beta : I \to X$

$(p \circ \beta)(0) = p(e_0) = x_0$

$(p \circ \beta)(1) = p(e_1) = x_0$ since $e_1 \in p^{-1}(x_0)$.

$p \circ \beta$ is a loop based at x_0.

$\Rightarrow [p \circ \beta] \in \pi_1(X, x_0)$.

$\varphi([p\circ\beta]) = (\widetilde{p \circ \beta})_{e_0}(1) = \beta(1) = e_1$. ✓

Therefore, φ is surjective.

- Now we will show that φ is injective:

Suppose $\varphi([\alpha]) = \varphi([\beta]) = e_1 \in p^{-1}(x_0)$, where $[\alpha], [\beta] \in \pi_1(X, x_0)$.

Let $\tilde{\alpha}$ be the lift of α and $\tilde{\beta}$ the lift of β.

Then $\tilde{\alpha}, \tilde{\beta}$ are paths in E from e_0 to e_1.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[ur, "\tilde{\beta}\ \tilde{\alpha}"] \arrow[r] & X \end{tikzcd}$$

$p \circ \tilde{\alpha} = \alpha$

$p \circ \tilde{\beta} = \beta$

Since E is simply connected, $\pi_1(E, e_0) = 0$.

Observe that $\tilde{\alpha} \cdot \tilde{\beta}^{-1} \in \pi_1(E, e_0)$ is a loop:

$\tilde{\alpha}(0) = \tilde{\beta}^{-1}(1) = e_0$

$\tilde{\alpha}(1) = \tilde{\beta}^{-1}(0) = e_1$

So we have that $\tilde{\alpha} \cdot \tilde{\beta}^{-1} \in \pi_1(E, e_0) = 0 \Rightarrow \tilde{\alpha} \cdot \tilde{\beta}^{-1}$ is homotopic to a constant loop, i.e., $\tilde{\alpha} \cdot \tilde{\beta}^{-1}$ is null-homotopic $\Rightarrow \alpha \cdot \beta^{-1}$ is null-homotopic.

(If H is the homotopy btwn $\tilde{\alpha} \cdot \tilde{\beta}^{-1}$ and a constant, then $p \circ H$ is the homotopy btwn $\alpha \cdot \beta^{-1}$ and a constant).

Therefore, $\alpha \cdot \beta^{-1}$ is homotopic to a constant loop.

$\Rightarrow [\alpha][\beta^{-1}] = 1 \Rightarrow [\alpha] = [\beta]$.

Thus, φ is injective.

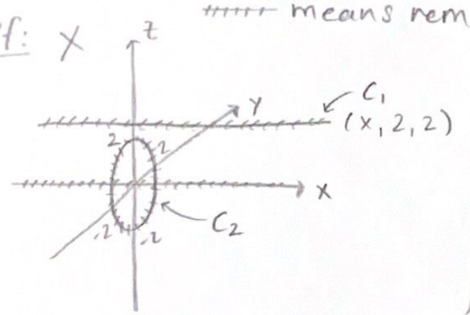
We conclude that φ is bijective. $\square$

...ed.

Let X be the space obtained from $\mathbb{R}^3$ by removing the x-axis, the straight line $C_1 = \{(x,2,2) : x \in \mathbb{R}\}$, and the circle $C_2 = \{(0,y,z) \in \mathbb{R}^3 : y^2+z^2 = 4\}$. Compute $\pi_1(X)$.

means removed

Pf: X



This is X, where the lines are removed from $\mathbb{R}^3$.

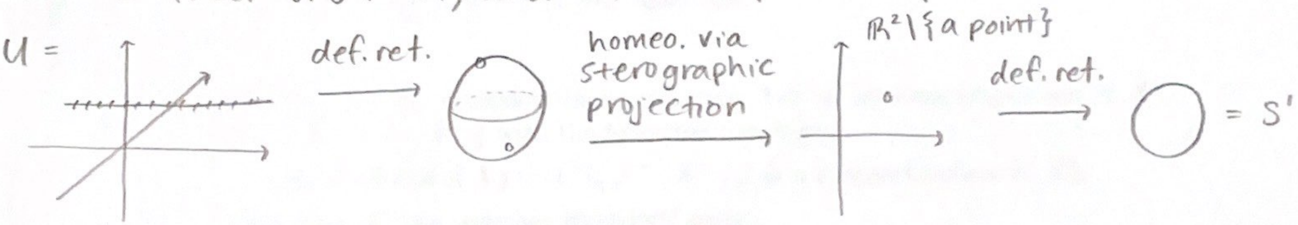
Observe that there is a positive distance between the circle C_2 and the line C_1.

Let $U = X \setminus \{(x,y,z) \in \mathbb{R}^3 : y+z > 3\}$ ← this has C_1 removed.

Let $V = X \setminus \{(x,y,z) \in \mathbb{R}^3 : y+z < 3.5\}$ ← this has C_2 and the x-axis removed.

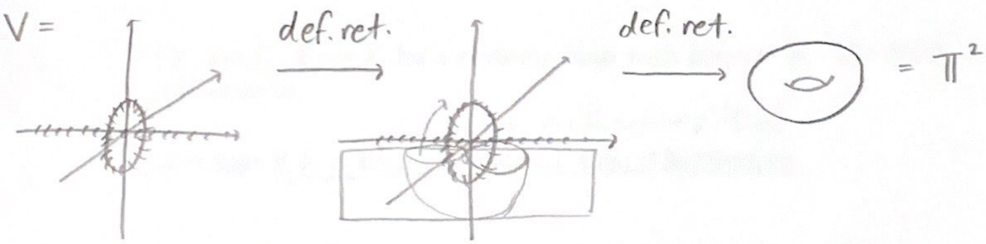
Observe that $U \cup V = X$, and U, V are open and path-connected.

- $U =$



So $\pi_1(U) = \pi_1(S^1) = \mathbb{Z}$.

- $V =$



So $\pi_1(V) = \pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

Observe that $U \cap V =$ a strip of $\mathbb{R}^3$ that does not have anything removed. So $U \cap V$ is path-connected and nonempty and a strip of $\mathbb{R}^3$ is convex, so $\pi_1(U \cap V) = 0$.

Since $U \cap V$ is simply connected, we can use the following version of Van-Kampen: $\pi_1(X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z})$.

Therefore, $\pi_1(X) = \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z})$.

$\square$