Topology Midterm 1

① True/False.

(a) A second countable Hausdorff space must be regular.

Pf. False.

Counter-example: Consider the space $(\mathbb{R}, \tau)$ where τ is the topology generated by the basis $\{(a,b), (a,b)\setminus K : a<b \in \mathbb{R}\}$, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$.

The space is second countable (just use $a,b \in \mathbb{Q}$).

The space is Hausdorff because it is $\mathbb{R}$ with a finer topology than the standard topology, and $(\mathbb{R}, \text{standard})$ is Hausdorff.

To see that $(\mathbb{R},\tau)$ is not regular, first notice that K is closed in $(\mathbb{R},\tau)$, since its complement is open: $\mathbb{R}\setminus K = \bigcup_{n\in\mathbb{N}} (-n,n)\setminus K$.

We claim that 0 cannot be separated from K by open sets.

Let U be any nbhd of 0 and V any open set containing K.

Then U contains a basis element of the form $(-\varepsilon,\varepsilon)\setminus K$.

Choose $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

V is open and contains $\frac{1}{n}$, so it contains a basis element containing $\frac{1}{n}$, which must have the form (a,b) with $a < \frac{1}{n} < b$.

Hence, $a < \varepsilon$.

Choose any irrational number ω in the interval $(a, \min\{\varepsilon, b\})$ and ω will be contained in both U and V, so $U \cap V \neq \emptyset$. □

(b) A compact metric space is second countable.

Pf. True.

A metrizable space is second countable iff it is Lindeloff iff it is separable.

If X is a compact metric space, then every open cover of X admits a finite subcover. This immediately implies that X is Lindeloff (which only asks for every open cover to admit a countable subcover).

A Lindeloff metric space is second countable. □

continued...

(c) A connected space is locally path-connected.

Pf: False.

Counter-example: Consider $\mathbb{R}^2$ with its standard topology and let K be the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. The set C defined by: $(\{0\} \times [0,1]) \cup (K \times [0,1]) \cup ([0,1] \times \{0\})$ considered as a subspace of $\mathbb{R}^2$ equipped with the subspace topology is known as the comb space.

The comb space is path-connected $\Rightarrow$ connected, but it is not locally path-connected. $\square$

(d) Let $\{A_\alpha\}_{\alpha \in I}$ be a family of subsets in a space X. Then $\overline{\bigcup_{\alpha \in I} A_\alpha} = \bigcup_{\alpha \in I} \overline{A_\alpha}$.

Pf: False. Could be uncountable (only holds for countable).

Counter-example: Let $X = \mathbb{R}$ and consider the subsets $\{A_n\}_{n=1}^{\infty} = \{(\frac{1}{n}, 1)\}_{n=1}^{\infty}$

Then $\overline{A_n} = [\frac{1}{n}, 1]$ for $n = 1$ to ∞.

So $\bigcup_{n=1}^{\infty} \overline{A_n} = (0, 1]$.

On the other hand, we have $\overline{\bigcup_{n=1}^{\infty} A_n} = [0, 1]$.

It is clear that $(0,1] \neq [0,1]$.

Therefore, we conclude that $\overline{\bigcup_{n=1}^{\infty} A_n} \neq \bigcup_{n=1}^{\infty} \overline{A_n}$ for $\{A_n\}_{n=1}^{\infty} \subseteq X$. $\square$

(e) Let f and g be two continuous maps from $\mathbb{R}$ with the standard topology to a topological space Y. Assume that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Pf: False. Y needs to be Hausdorff.

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Let $\mathbb{RP}^2$ be the real projective plane, i.e., the quotient space $(\mathbb{R}^3\setminus\{0\})/\sim$ where $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if and only if $(x_0, x_1, x_2) = \lambda(y_0, y_1, y_2)$ for some $\lambda \in \mathbb{R}\setminus\{0\}$.
Let $\pi : \mathbb{R}^3\setminus\{0\} \to \mathbb{RP}^2$ be the quotient map, that is, $\pi(x_0, x_1, x_2)$ is the equivalent class $[x_0, x_1, x_2]$ in $\mathbb{RP}^2$.

(1) Given $E = \{(0, x_1, x_2) \in \mathbb{R}^3;\ x_1^2 + x_2^2 = 1\}$, explicitly write down $\pi^{-1}(\pi(E))$ in $\mathbb{R}^3\setminus\{0\}$.

Pf: We have that $\pi(E) = \{[(0, x_1, x_2)] : (0, x_1, x_2) \in \mathbb{R}^3,\ x_1^2 + x_2^2 = 1\}$, where $[(0, x_1, x_2)]$ denotes the equivalence class of the point $(0, x_1, x_2)$ in $\mathbb{RP}^2$.

Then for each $[(0, x_1, x_2)] \in \pi(E)$, we have

$$\begin{aligned}\pi^{-1}(\{[(0,x_1,x_2)]\}) &= \{(y_0, y_1, y_2) \in \mathbb{R}^3 : (y_0, y_1, y_2) = \lambda(0, x_1, x_2),\ \lambda \in \mathbb{R}\setminus\{0\}\} \\ &= \{(0, y_1, y_2) \in \mathbb{R}^3 : (0, y_1, y_2) = (0, \lambda x_1, \lambda x_2),\ \lambda \in \mathbb{R}\setminus\{0\}\}.\end{aligned}$$

Using the original restrictions on x_1 and x_2, this becomes

$$\pi^{-1}(\pi(E)) = \{(0, y_1, y_2) \in \mathbb{R}^3\setminus\{0\}\} = y,z \text{ plane without the origin.}$$

$\square$

(2) Is $\mathbb{RP}^2$ path-connected? Prove your assertion.

Pf: We have the quotient map $\pi : \mathbb{R}^3\setminus\{0\} \to \mathbb{RP}^2$.
Notice that $\mathbb{R}^3\setminus\{0\}$ is clearly path-connected.
The continuous image of a path-conn. set is path-conn.
Since π is a quotient map, it is continuous, and since $\mathbb{R}^3\setminus\{0\}$ is path-conn., we have that $\pi(\mathbb{R}^3\setminus\{0\})$ is path-conn.
Since π is a quotient map, it is surjective, so $\pi(\mathbb{R}^3\setminus\{0\}) = \mathbb{RP}^2$.
Therefore, we conclude that $\mathbb{RP}^2$ is path-connected.

$\square$