Topology Midterm 2

① True/False.

(a) The connected sum $M \# N$ of two connected 2-dimensional manifolds M and N satisfies $\pi_1(M \# N) = \pi_1(M) * \pi_1(N)$.

Pf: False. (It would be true for n-dim. manifolds, where $n \geq 3$).

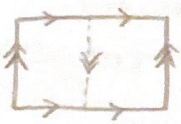
Counter-example: let $M = N = \mathbb{RP}^2$.

We have that $\mathbb{RP}^2 \# \mathbb{RP}^2 = K$, where K is the klein bottle.

Then $\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) * \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, whereas $\pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$, which has no element of order 2, so it is not isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. $\square$

(b) There is a covering map from the torus $\mathbb{T}^2 = S^1 \times S^1$ to the Klein bottle.

Pf: Observe that if we take a torus and draw a line in the middle as follows, we have two Klein bottles.



(True)

$\square$

(d) Let $\tilde{X}$ be a compact space, let X be a path-connected space, and let $p: \tilde{X} \to X$ be a covering map. Given any $x \in X$ and $\tilde{x} \in p^{-1}(x)$, the number $\# p^{-1}(x)$ of sheets covering must be equal to the index $[\pi_1(X, x) : p_*(\pi_1(\tilde{X}, \tilde{x}))]$ of the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$.

Pf: False, because $\tilde{X}$ need not be path-connected.

Counter-example: Let $X = S^2$ and $\tilde{X} = S^2 \times Y$, where Y is the set $\{0, 1\}$ equipped with the discrete topology. (We can think of $\tilde{X}$ as the disjoint union of two spheres).

Then the projection map $p: \tilde{X} \to X$ given by $(x, i) \mapsto x$ is a covering map, but the cardinality of each fiber is 2, and since $\pi_1(X)$ is trivial, the index of the subgroup is 1. $\square$

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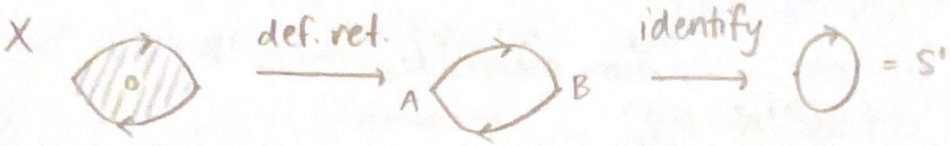
(e) Let X be $\mathbb{RP}^2$ with one point removed. Then, $\pi_1(X) = \mathbb{Z}$, where "=" stands for the group isomorphism.

Pf: True.



(just the perimeter)

$\pi_1(X) = \pi_1(S^1) = \mathbb{Z}$. $\square$

(c) Let $f: S^2 \to S^2$ be a continuous map such that $f(w) \neq -w$ for any $w \in S^2$. Then, the map f must be homotopic to the identity map on S^2.

Pf: True.

Let $id_{S^2}: S^2 \to S^2$ given by $id_{S^2}(w) = w$.

Let $H: [0,1] \times S^2 \to S^2$ be given by $H(t, w) = \dfrac{(1-t)f(w) + t \cdot id_{S^2}(w)}{\|(1-t)f(w) + t\, id_{S^2}(w)\|}$.

Observe that H is cts since it's the product and sum of cts. fns.

First we will show that H is well-defined, i.e., $(1-t)f(w) + t\, id_{S^2}(w) \neq 0$:

$(1-t)f(w) + t \cdot id_{S^2}(w) = 0 \Rightarrow (1-t)f(w) = -t\, id_{S^2}(w)$

$\|(1-t)f(w)\| = \|-t \cdot id_{S^2}(w)\|$

since $\|f(w)\| = 1$, $\|id_{S^2}(w)\| = 1$ $\longrightarrow$ $\|(1-t)\| = \|-t\|$

$1-t = t$

$1 = 2t$

$\Rightarrow t = \frac{1}{2}$

Plugging in $t = \frac{1}{2}$, we get: $(1-\frac{1}{2})f(w) = -\frac{1}{2} id_{S^2}(w)$

$\frac{1}{2} f(w) = -\frac{1}{2} id_{S^2}(w)$

$f(w) = -1 id_{S^2}(w)$

$f(w) = -w$,

which cannot happen since $f(w) \neq -w$ for any $w \in S^2$.

Therefore, H is well-defined.

Now we will check that H is a homotopy:

...ued...

$$H(0,w) = \frac{(1-0)f(w) + 0 \cdot id_{S^2}(w)}{\|(1-0)f(w) + 0 \cdot id_{S^2}(w)\|} = \frac{f(w)}{\|f(w)\|} = f(w), \text{ and}$$

$$H(1,w) = \frac{(1-1)f(w) + 1 \cdot id_{S^2}(w)}{\|(1-1)f(w) + 1 \cdot id_{S^2}(w)\|} = \frac{id_{S^2}(w)}{\|id_{S^2}(w)\|} = id_{S^2}(w) = w.$$

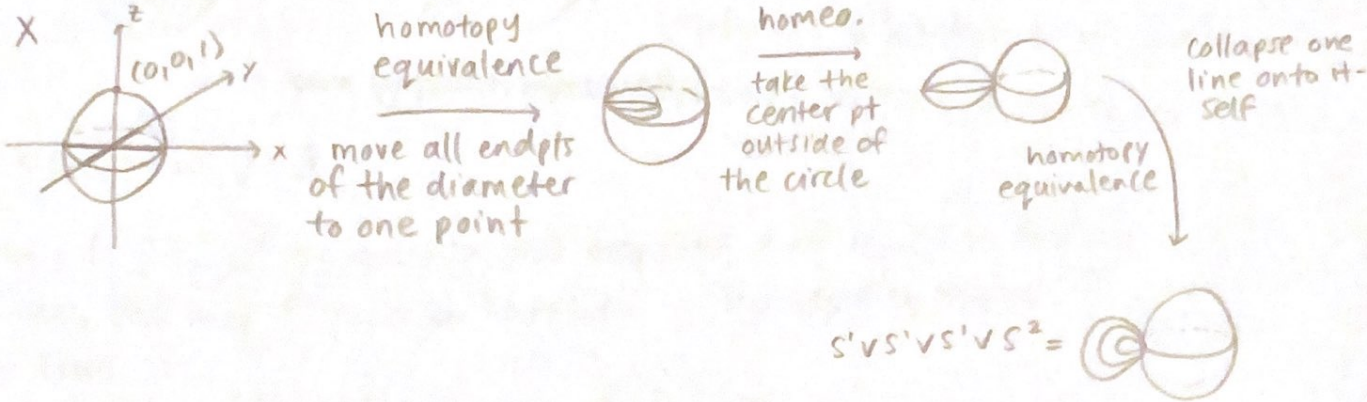
Therefore, H is indeed a homotopy.

Thus, the map f must be homotopic to the identity map on S^2. $\square$

continued...

② Let $X \subset \mathbb{R}^3$ be the union of the unit sphere S^2 with the line segments $\{(x,y,0) \in \mathbb{R}^3;\ xy = 0,\ x^2+y^2 \leq 1\}$. Compute $\pi_1(X, x_0)$ with $x_0 = (0,0,1)$.

Pf: Since X is path-connected, the fund. gp. is independent of the base point up to isomorphism.



Each space S^1 and S^2 is locally Euclidean, so the wedge point has a nbhd in each space that deformation retracts to the wedge point. So using Van-Kampen, we can take the fund. gp. as follows:

$$\begin{aligned}\pi_1(X) &= \pi_1(S^1 \vee S^1 \vee S^1 \vee S^2)\\ &= \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^2)\\ &= \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * 0\\ &= \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.\end{aligned}$$

Therefore, $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. $\square$